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# Group Theoretic Model Interactions as an Approach to 2s-1d Shell Nuclei

## The Level Structure of Fluorine-20

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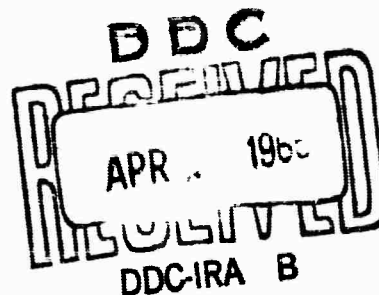
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## ABSTRACT

Based on the studies of S. Lie and E. Cartan in the theory of continuous groups of transformations, M. Meshinsky has reformulated the nuclear shell many-body problem in second-quantization language. The methods due to J.P. Elliott for simplifying the basis set of state functions by classification according to the group  $SU(3)$  are recast into the above-mentioned reformulation. The purpose is to make low-energy nuclear calculations feasible for nuclei with 4 and more particles in the 2s-1d shell and thus render the possibility of probing for  $SU(3)$  symmetries in these nuclei. A hamiltonian model consisting of pairing and quadrupole-quadrupole terms is known to approximate respectively the short- and long-ranged correlations between nucleons given by an arbitrary, reasonably shaped two-nucleon central interaction potential. The former model is generalized to include exchange effects at the long range as well as spin-orbit coupling, and is studied in detail from the viewpoint of its various group symmetries. It is then employed to calculate the low-lying levels of Fluorine-20 which show reasonable accord with the empirical level-scheme.

## PROBLEM STATUS

This is an interim report on this problem; work is continuing.



## AUTHORIZATION

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## I. INTRODUCTION

Angular momentum techniques have proven extremely useful in simplifying the treatment of numerous quantum-mechanical problems in atomic, molecular, nuclear and solid state physics. These techniques, first developed in the early 1930's, involve such concepts as rotation matrices, spherical harmonics, vector-coupling, recoupling by Racah coefficients, 9-j coefficients transforming an L-S to a j-j scheme, irreducible tensor operators and state functions, etc. The validity of such techniques rests of course on the rotational invariance of many physical situations. Group theoretically speaking, the hamiltonian of a spherically symmetric problem commutes with the generators of the group of rotations in a 3-dimensions,  $R_3$ .

The fact that a large class of problems in addition possess symmetry groups larger than  $R_3$  can be exploited by studying techniques similar to angular momentum methods but generalized to definite groups containing  $R_3$  as a subgroup. Solving the Schrödinger equation of a many-body problem, even approximately, is a formidable if not impossible task. The matrix mechanical approach consisting essentially in setting up and diagonalizing the hamiltonian matrix is more promising and adaptable to the utilization of these higher symmetries as then the original matrix is decomposed into smaller sub-matrices. Even considering the capacity of modern electronic computers to diagonalize large matrices, the labor involved in calculating the elements of the sub-matrices is still monstrous due to the very large number of N-particle states present. It thus becomes desirable to formulate the problem in terms more easily adapted to computer languages so that the machine can do more than merely diagonalize matrices.

Consider the asymmetric top hamiltonian

$$H_T = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3} \quad (1)$$

where  $L_1, L_2, L_3$  are the operator angular momentum components in a system fixed in the body, and the  $J$ 's the three Cartesian moments of inertia. Solving the problem for the allowable energies entails construction of the matrix  $\|H_T\|$  in some adequate basis, e. g., the set  $|lm\rangle$  of eigenfunctions of operators  $L^2$  and  $L_3$ . Thus, since  $L^2|lm\rangle = l(l+1)|lm\rangle$  and  $L_3|lm\rangle = m|lm\rangle$  while  $L^2 = L_1^2 + L_2^2 + L_3^2$ , our hamiltonian simplifies to

$$H_T = \left(\frac{1}{2J_2}\right)L^2 + \left(\frac{1}{2J_3} - \frac{1}{2J_2}\right)L_3^2 + \left(\frac{1}{2J_1} - \frac{1}{2J_2}\right)L_1^2 \quad (2)$$

the first two terms being diagonal in the  $|lm\rangle$  basis set; we notice that  $H_T$  is quadratic in the non-diagonal operator  $L_1$ . Our matrix  $\|H_T\|$  whose rows and columns are given by all the allowed values of  $l$  and  $m$  is

$$\| \langle l m' | H_T | l m \rangle \| =$$

$$\left\| \left\{ \frac{l(l+1)}{2J_2} + \frac{1}{2} \left( \frac{1}{J_3} - \frac{1}{J_2} \right) m^2 \right\} \delta_{m'm} + \frac{1}{2} \left( \frac{1}{J_1} - \frac{1}{J_2} \right) \langle l m' | L_1^2 | l m \rangle \right\| \quad (3)$$

where, since matrix elements between different  $l$ -values vanish, the complete matrix is now decomposed into as many blocks of elements as there are different  $l$ -values in the problem: the rows and columns within each block being labelled by  $m$ , where  $-l \leq m \leq l$  and  $l$  designates the whole block. Moreover, the complete solution of this exact calculation of energies requires only knowledge of the matrix elements

$$\langle l m' | L_1^2 | l m \rangle \quad (4)$$

But these are definitely obtainable in closed algebraic form by simple angular momentum techniques based solely on the commutation relations

$$[L_1, L_2] = -i L_3 \quad (\text{and cyclically}) \quad (5)$$

The solution is well known, but no recurrence to its explicit form is necessary -- only the simple relations (5) are needed to obtain

$$\langle l m' | L_1 | l m \rangle = \frac{1}{2} \sqrt{(l-m)(l+m+1)} \delta_{m', m+1} + \frac{1}{2} \sqrt{(l+m)(l-m+1)} \delta_{m', m-1} \quad (6)$$

and thus solve (3) exactly.

In group theoretic language, the generators of infinitesimal rotations around the 1, 2 and 3 axes are  $L_1$ ,  $L_2$  and  $L_3$  -- called simply the  $R_3$  group generators. Commutation relations between them (5) form a Lie algebra. The Operator  $L^2$  -- formed out of the group generators as the sum of their squares, commutes with all three generators  $L_1$ ,  $L_2$ ,  $L_3$  and its eigenvalue  $\ell(\ell+1)$  sufficing to characterize the  $R_3$  transformation properties of  $|\ell m\rangle$  -- is the Casimir operator of  $R_3$  whose eigenvalue provides the classification label  $\ell$ . The rows of each irreducible representation designated by  $\ell$  are specified by another label  $m$  which proceeds from the  $R_2$  (subgroup of  $R_3$ ) Casimir operator  $L_3$ . The set of functions  $|\ell m\rangle$  is thus said to transform irreducibly under  $R_3$  and explicitly also under its subgroup  $R_2$ .

Moshinsky's group-theoretic interpretation of second-quantization techniques applied to the many-body problem lead to straightforward generalizations of these simple  $R_3$  group results to the case of physical problems involving larger symmetries associated with permutations, the harmonic oscillator common potential,  $r$ -dimensional rotations and spin-isospin. The nuclear shell model problem with a spherical (inert closed-shell) core of nucleons is given by the  $N$ -extra-shell nucleon hamiltonian

$$H = \sum_{i=1}^N \{T_i + U_i\} + \sum_{i < j=1}^N V_{ij} \quad (7)$$

where  $T_i = p_i^2/2m$  is the kinetic energy,  $U_i$  a central or non-central (or both) single-body interaction and  $V_{ij}$  a central two-body interaction. In the Moshinsky formulation, a single-body interaction operator is expressible as a linear combination, and a two-body interaction operator as a bilinear combination, of the generators belonging to groups of

symmetries higher than  $R_3$ . But these generators form a known commutator Lie algebra. Therefore, in principle, closed expressions for the matrix elements of any single- and two-body interaction could be obtained.

In practice, however, another approach was found both simpler and physically more meaningful: to consider the problem of allowable energies associated with a mixture of pairing and quadrupole-quadrupole interactions as a model for central two-body interaction, plus a spin-orbit single-body interaction. Contrary to the concept of a potential, these model interaction operators have a clear group-theoretic meaning in that they can be written in terms of operators which commute with the generators of various related groups and whose eigenvalues serve as irreducible representation labels. These operators are none other than the Casimir operators of the group involved. Thus the irreducible basis sets diagonalizing separately the three interactions mentioned above could be constructed by elementary algebraic techniques based on simple notions from group theory. Having chosen one of the three sets, the nondiagonal matrices of the other two interaction operators can be constructed in this base.

We chose the quadrupole-quadrupole (resembling the long-ranged part of a central two-body residual interaction) scheme for three reasons: (1) it is invariant under the group  $U_3$ , the algebraic techniques of which have been studied extensively by



Elliott, Biedenharn, Moshinsky and others; (2) there is a close resemblance between the  $U_3$  or  $SU_3$  scheme and the states of the collective rotational nuclear model. There seems also to be some connection between this interaction operator and quadrupolar nuclear vibrations; and (3) classification of states by  $SU_3$  offers the possibility suggested by Elliott of restricting, as a first approximation, the basis to the single  $SU_3$  representation which lies lowest in energy.

These methods should be extremely powerful to carry out, within a feasible length of time, calculations on families of nuclei with the aim of making global studies of their diverse properties. The isotope Fluorine-20 is merely a "pilot nucleus" for our work; the fact that it is odd-odd makes it a more difficult shell model problem as such, in spite of having only four nucleons outside the doubly-magic  $^{16}_8\text{O}_8$  core. Little is known empirically about the low-lying spins of this nucleus unambiguously, but our results on the whole are not inconsistent with known experiments to date.

## II. NUCLEAR MODELS & LOW-ENERGY NUCLEAR STRUCTURE.

The nuclear shell model of Mayer & Jensen, with residual interactions between extra-closed-shell nucleons has been widely applied<sup>1, 2)</sup> to account for such low-energy nuclear properties as level energies, spins and parities, moments, electromagnetic and  $\beta$ -decay transition rates and even binding energies. Its successes have been encouraging but glaring failures are evident in some respects, viz., large quadrupole moments are left unexplained.

A second approach to the problem has grown from evidence of cooperative nucleonic behavior seen in the fission process and the partial success of the liquid-drop model which seems to be the antithesis of shell structure. The liquid-drop and shell-model viewpoints were combined by Bohr & Mottelson and the Copenhagen school to propose a nuclear model allowing more generalized motion within the nucleus by the introduction of collective vibrational and rotational degrees of motion.

A third trend has been to return to the shell model but with specific residual interaction models that simulate those collective aspects to a certain extent. So much the better if these model residual interaction possess group symmetries which can be systematically exploited to reduce calculational labor.

### 1. Heavy Nuclei

1a. Collective Behavior. In 1950 Rainwater<sup>3, 4)</sup> suggested the possibility of nuclei between magic proton and neutron numbers of acquiring equilibrium non-spherical shapes to account for observed large quadrupole moments and transition rates ---- as much as a single nucleon outside a closed-shell core having the power to polarize or deform the core by centrifugal forces. Bohr and Mottelson<sup>5, 6</sup> proposed (1952-3) a unified description whereby

shell structure due to the particles was maintained alongside collective structure consisting of permanent deformations resulting in observed rotation-like spectra as well as vibrations in size and shape. They put forward a total hamiltonian

$$\mathcal{H} = H_p + H_s + H_{int} \quad (8)$$

with particle-like  $H_p$ , surface effects  $H_s$  arising from deviations from perfect sphericity and an interaction  $H_{int}$  between particle and surface effects. For a small number of extra-closed-shell nucleons one had weak-coupling since departure from sphericity was small enough that one could treat  $H_s + H_{int}$  in a perturbation method with shell model particle states  $\chi$  as zero-order functions. For many particles strong-coupling prevailed, that is, spheroidal shapes occurred permitting a reasonable description in terms of well-known rotating-top eigenfunctions  $D_{int}$ , with perturbative effects due to vibration, vibration-rotation and surface-particle couplings. The Unified Model enjoyed great success<sup>7)</sup> in certain definite regions of the nuclide table, failing seriously in others mainly because of the difficult intermediate coupling situations.

Nilsson<sup>8)</sup> (1955), using the model of a single-particle in an anisotropic harmonic oscillator common potential with axial symmetry plus a single-body spin-orbit term and a term in  $\epsilon^2$  to simulate the partial effect of a square-well, calculated for every nuclear oscillator shell the single-particle energy levels as functions of a parameter proportional directly to quadrupolar oscillator deformation and inversely to spin-orbit strength. Many applications of this simple model have been made to odd-mass nuclei, where the even

number outside closed-shells are presumed to pair off according to the shell model, leaving the one odd particle to deal with. Numerous properties were predicted approximately<sup>9, 10)</sup> in those regions of strong or extremely weak deformation where the model is expected to apply but again, failing for intermediate cases.

Both the Bohr-Mottelson and Nilsson models have succeeded in explaining, within certain limits, the rotational band-like level structure of many strongly deformed nuclei beyond  $A = 150$ . These bands are designated by a quantum number  $K$  which represents the projection on the nuclear symmetry axis of the total angular momentum  $J$  composed of individual-particle  $j$  and collective  $R$  angular momenta, as shown in figure 1.

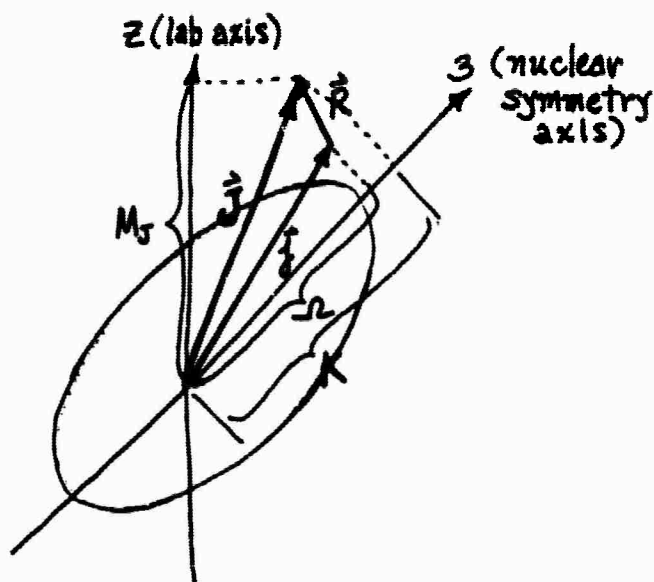


FIGURE 1

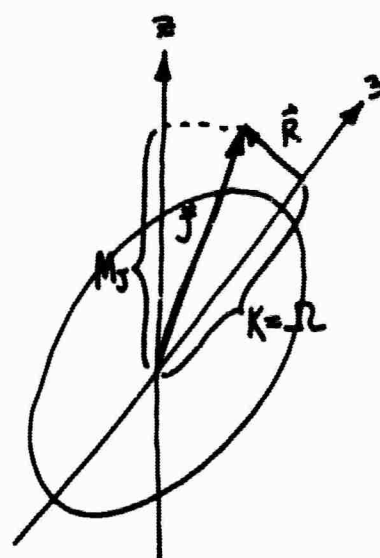


FIGURE 2

For low energies, rotations are about axes perpendicular to the nuclear symmetry axis 3 so that the total individual-particle angular momentum vector  $\vec{J}$  coincides with the 3-axis,  $\Omega$  becoming equal to K, and one has the situation in Figure 2. The nuclear wave function referred to this axis is then

$$\chi_K = \chi_\Omega = \frac{1}{\sqrt{N!}} \sum_P (-)^P \chi_{\Omega_1} \chi_{\Omega_2} \cdots \chi_{\Omega_N}, \quad (9)$$

or the antisymmetrized product of N individual-particle wave functions.

Perhaps with the intention of having a model capable of covering a wider range of cases than the rotational models discussed above and the original Mayer-Jensen shell model, a new approach has become very popular since 1957. Elliott<sup>10,11</sup>) Bohr and Mottelson discussed the use of long-and short-range interactions approximated respectively by a quadrupole-quadrupole ( $Q^2$ ) and a pairing P force. The  $Q^2$  force to be distinguished from a single-particle Q force of the form

$$\sum_{i=1}^N r_i Y_{20}(\theta_i, \varphi_i) \quad (10)$$

used by Nilsson to deform the common oscillator well ---- it is a two-body interaction between extra-closed shell nucleons giving rise to what Mottelson called the "aligned coupling scheme" whereby the extra-shell nucleon orbits tend to align themselves along a given axis fixed in the core thus acting, effectively, as if the core itself were deformed into a (quadrupolar) ellipsoidal shape. Mottelson<sup>12)</sup> shows that  $P$  gives rise to binding energies depending on  $N$ , while  $Q^2$  produces effects dependent on  $N^2$  and therefore for many particles outside the closed shells  $Q^2$  is expected to predominate.

1b. Pairing Effects. Bohr, Mottelson & Pines<sup>13)</sup> (1958) emphasized that the large spacing (gap) between the ground and first excited states of even-even nuclei as well as the mass difference between even-even and odd  $A$  nuclei may be indicative of nucleonic pairing.

The first formal use of the  $P$  plus  $Q^2$  model seems to be due Belyaev<sup>14)</sup> (1959) who reached very interesting conclusions regarding the effect of pairing in heavy nuclei: 1) pairing reduces the heretofore too-large predicted nuclear moments of inertia by magnitudes in much better accord with experiment. (Griffin & Rich<sup>15)</sup> (1960); Nilsson & Prior<sup>16)</sup> (1960) verified this admirably for 26 even-even rare earth nuclei with an average theory-to-experiment difference of only 6%). 2) Near closed shells, pairing gives rise to spherical equilibrium shapes while the  $Q^2$  interaction low-energy vibrational modes is responsible with frequencies within observed trends. 3) The even-even nuclei gap is explained, as well as the increased level density above it. A disadvantage of Belyaev's 2nd - quantization treatment of pairing is

that the number of particles  $N$  is not constant so that results apply to average properties in a given isotope region.

The Belyaev model was used by Kisslinger & Sorensen<sup>17)</sup> (1960) in an extensive application to single-closed-shell heavy nuclei, for cases where  $P$  predominates over  $Q^2$ , and obtained generally encouraging results of level energies, moments and transition rates. A recent work by these authors<sup>18)</sup> (1963) shows use of the same model with the additional treatment of quadrupolar vibrations via introduction of the phonon formalism. Reasonable agreement with empirical low-energy systematics is obtained for numerous heavy nuclei outside well-established regions of nuclear deformation.

## 2. Light Nuclei.

Experimental work by Litherland et al<sup>19)</sup> (1956) on  $Mg^{24}$  strongly suggested the presence of rotational (collective) structure in the  $A=24,25$  mass region of light nuclei. Subsequent experiments corroborated this suggestion for other nuclei in the 2S-1d shell, e.g.,  $F^{19}$  and  $N_e^{20}$ .

The theoretical structure of  $F^{19}$  presented a curious situation: two apparently very different models yielded very similar results<sup>20)</sup>. The results obtained by Elliott & Flowers<sup>21)</sup> (1955) with a central Yukawa interaction and spin-orbit force acting on mixed shell model configurations and those of Paul<sup>22)</sup> (1957) using the Nilsson model showed that, at least for this nucleus, the two models could not be very distinct. This embarked Elliott on a series of key researches leading to his classic 1958 papers<sup>11)</sup>. He found that collective deformation with its associated rotational spectra is obtainable by considering particles in a harmonic oscillator

common potential and interacting with a two-body force of angular dependence  $P_2(\cos \theta)$ , i.e., the  $Q^2$  force, which is diagonal in an  $SU_3$  basis of mixed-configurational states ----  $SU_3$  refers to the group of unimodular unitary transformations in three dimensions. The  $SU_3$  states referred to laboratory axes are characterized by definite orbital angular momentum  $L$  and projection  $M$ , as well as by an approximate quantum number  $K$  (appearing to be related to the rotational band quantum number  $K$  of Figure 2), the  $SU_3$  irreducible representation  $(\lambda\mu)$  and orbital permutation symmetry  $[f]$ . These  $\Psi([f](\lambda\mu) KLM)$  are projected out of intrinsic functions  $\chi([f](\lambda\mu) K)$  referred to a nuclear axis. The  $\chi$  functions are classified by the subgroup  $U_2$  in addition to  $SU_3$  and are eigenfunctions of an anisotropic axially symmetric harmonic oscillator potential. For  $N=2,3,4$  particles in the 2s-1d unfilled shell he found good overlaps between L-S coupled shell model wave functions and his  $SU_3$  basis set of states corresponding to the leading (lowest-energy) representation of  $SU_3$ .

For nuclei with a few particles in the 1-p shell Kurath & Picman<sup>23)</sup> (1959) found strong overlaps between wave functions constructed by the Elliott  $SU_3$  technique applied to Nilsson intrinsic states and shell model intermediate coupling ( $jj$  and  $LS$ ) wave functions from calculations<sup>24)</sup> with a two-body central force in the limit of zero spin-orbit force. Similar results for nuclei at the beginning of the 2s-1d shell were found by Redlich.<sup>25)</sup>

The Elliott  $SU_3$  technique was applied extensively to nuclei in the 1p shell ---- with the inclusion of spin-orbit interaction ---- by Koltun<sup>26)</sup> (1961) with results comparing favorably with earlier intermediate coupling calculations by Kurath<sup>25)</sup> (1956).



These techniques were extended to the 2s-1d shell by Banerjee & Levinson<sup>27)</sup> (1963). Calculations on  $Mg^{24}$  (N=8 in 2s-1d shell) were carried out by Elliott & Harvey<sup>78)</sup> who found a small (10 to 20%) mixing of other  $SU_3$  representations into the leading one for this nucleus. Chacón & Moshinsky<sup>29)</sup> (1962) calculated the low-lying levels of  $Ne^{20}$  using a competitive mixture of P and  $Q^2$  forces and, separately, under a gaussian central potential. A remarkable resemblance between the two predicted level-schemes emerged, as well as excellent agreement for the very lowest excited levels. The  $Ne^{20}$  E2 transition lifetimes were calculated on this model by the author and co-workers<sup>30)</sup> (1963) showing the tendency of  $Q^2$  to deform states (decrease lifetimes) and P to produce more spherical states (increase lifetimes).

The method to be used in our work will be within the third approach mentioned in the beginning of this chapter. Thus, our model will comprise P,  $Q^2$  and spin-orbit interactions whose group symmetries shall be employed to advantage. There seems to be no "a priori" reason why this interaction hamiltonian model (involving a very small number of parameters) should be restricted to certain regions of the table of nuclides.

CHAPTER

III.- THE MOSHINSKY GROUP THEORETICAL REFORMULATION OF THE NUCLEAR SHELL MANY - BODY PROBLEM.<sup>31)</sup>

1. Creation and Annihilation Fermi Operators.

A single particle in a state  $\rho$  is defined by a creation operator acting on a vacuum state, namely

$$b_{\rho}^{+} |0\rangle$$

which corresponds to  $\psi_{\rho}(\mathbf{r})$ . An annihilation operator is given simply by the contravariant operator  $b^{\rho}$  where

$$b^{\rho} |0\rangle \equiv 0. \quad (11)$$

They obey the Fermi anti-commutation relations

$$\{b_{\rho}^{+}, b^{\rho'}\} = b_{\rho}^{+} b^{\rho'} + b^{\rho'} b_{\rho}^{+} = \delta_{\rho}^{\rho'} \quad (12a, b, c)$$

$$\{b_{\rho}^{+}, b_{\rho'}^{+}\} = \{b^{\rho}, b^{\rho'}\} = 0$$

and hence, expanding the left-hand side of the second relation

$$\{b_{\rho}^{+}, b_{\rho'}^{+}\} = b_{\rho}^{+} b_{\rho'}^{+} + b_{\rho'}^{+} b_{\rho}^{+} = 0$$

$$b_{\rho}^{+} b_{\rho'}^{+} = -b_{\rho'}^{+} b_{\rho}^{+}$$

it becomes obvious that if  $\rho = \rho'$ ,  $b_{\rho}^{+} b_{\rho}^{+} = 0$ , i.e., the Pauli principle is satisfied: one and only one particle can be in the state  $\rho$ . For particles in a common central potential the

state  $\rho$  is given by the assembly

$$\rho \rightarrow \nu l m, \sigma \tau \quad (14)$$

where  $\nu$  is the principal quantum number,  $l$  the orbital angular momentum,  $m$  the magnetic quantum number,  $\sigma$  the spin projection along an arbitrary axis and  $\tau$  the isotopic spin projection.

The allowed values of  $\rho$  thus label the dimensions of a single-particle total space composed of a coordinate (or orbital) subspace ( $\nu l m$ ) and a spin-isospin subspace ( $\sigma \tau$ ). The  $\rho$ -dimensional vector defining this space is  $b_{\rho}^{\dagger}$ . An equivalent assembly of quantum numbers for the single-particle state could be

$$\rho \rightarrow \nu l j m_j, \tau \quad (15)$$

where  $j$  results from coupling  $\vec{l}$  with  $\vec{s}$  and  $m_j$  is its projection. This choice is more appropriate for  $j-j$  coupling and here the single-particle space is decomposed into "spin-orbital" ( $\nu l j m_j$ ) and "isotopic spin" sub-spaces.

## 2. State of $N$ -noninteracting fermions.

A state of  $N$  non-interacting fermions is usually given by the normalised Slater determinant of single particle functions which is totally anti-symmetric under particle exchange:

$$\Phi \equiv \frac{1}{\sqrt{N!}} \sum_P (-)^P P \psi_{e_1}(1) \psi_{e_2}(2) \dots \psi_{e_N}(N) \equiv \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_{e_1}(1) & \psi_{e_1}(2) & \dots & \psi_{e_1}(N) \\ \psi_{e_2}(1) & \psi_{e_2}(2) & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{e_N}(1) & \dots & \dots & \psi_{e_N}(N) \end{vmatrix} \quad (16)$$

where  $P$  is a permutation of two variables from the set  $1, 2, \dots, N$ .

Taking another function

$$\Phi' \equiv \frac{1}{\sqrt{N!}} \sum_P (-)^P P \psi_{p'_1}(1) \psi_{p'_2}(2) \dots \psi_{p'_N}(N) \quad (17)$$

and considering the scalar product between  $\Phi$  and  $\Phi'$  one obtains the well-known result

$$(\Phi', \Phi) = \begin{vmatrix} \delta_{p_1}^{p'_1} & \delta_{p_1}^{p'_2} & \dots & \delta_{p_1}^{p'_N} \\ \delta_{p_2}^{p'_1} & \delta_{p_2}^{p'_2} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{p_N}^{p'_1} & \dots & \dots & \delta_{p_N}^{p'_N} \end{vmatrix} \equiv \delta_{p_1 p_2 \dots p_N}^{p'_1 p'_2 \dots p'_N} \quad (18)$$

which is called a "generalized delta function".

Now, since  $\psi_{p_i}$  corresponds to  $b_{p_i}^+ |0\rangle$  in second-quantization formulation it is natural to assume as the equivalent of  $\Phi$  the expression

$$|p_1 p_2 \dots p_N\rangle \equiv b_{p_1}^+ b_{p_2}^+ \dots b_{p_N}^+ |0\rangle \quad (19)$$

which from the anti-commutation relation (12b) is clearly anti-symmetric under particle exchange and therefore also satisfies the exclusion principle. The scalar product of (19) with another state  $|p'_1 p'_2 \dots p'_N\rangle$  can easily be shown to give

$$\begin{aligned} \langle p'_1 p'_2 \dots p'_N | p_1 p_2 \dots p_N \rangle &= \langle 0 | b^{p'_N} \dots b^{p'_2} b^{p'_1} b^+_{p_1} b^+_{p_2} \dots b^+_{p_N} | 0 \rangle \\ &= \delta_{p'_1 p_1 p'_2 p_2 \dots p'_N p_N} \end{aligned} \quad (20)$$

using the relations (12) and the fact that  $b^0 | 0 \rangle = 0$ , and this is identical with the result (18) for  $(\Phi', \Phi)$ .

### 3. Single-and Two-Body Operators as Linear and Bilinear Combinations of Group Generators.

An N-particle state with interactions, specifiable by a set of N-particle quantum numbers  $\lambda$ , can always be given as a superposition of linearly independent states of N-noninteracting-particles

$$P_\lambda = \sum_{p_1 p_2 \dots p_N} B^\lambda_{p_1 p_2 \dots p_N} b^+_{p_1} b^+_{p_2} \dots b^+_{p_N} | 0 \rangle \quad (21)$$

with arbitrary coefficients  $B^\lambda_{p_1 p_2 \dots p_N}$ . These coefficients will be determined by the operator set  $\Theta$  representing a complete set commuting of/observables required to characterize the polynomial base (21) exhaustively via the eigenvalue equations

$$\Theta P_\lambda = \theta^\lambda P_\lambda. \quad (22)$$

It is desirable therefore to obtain general operators which depend on the creation and annihilation operators  $b^+_{p'}$  and  $b^{p'}$ . The most common of operators are symmetric one-and two-body

scalar operators defined as

$$\sum_{i=1}^N W(r_i) \text{ and } \sum_{i,j=1}^N V(r_{ij}) \quad (23)$$

where  $r_i$  stands for all the coordinates of the  $i^{\text{th}}$  particle and  $r_{ij} \equiv |\vec{r}_i - \vec{r}_j|$ . Let us call  $W(r_i) \equiv W_i$  and  $V(r_{ij}) \equiv V_{ij}$  and their matrix elements in the usual form are

$$\langle \eta_i | W_i | \eta'_i \rangle = \int \psi_{\eta_i}^*(i) W_i \psi_{\eta'_i}(i) d\tau_i$$

$$\langle \eta_i \eta_j | V_{ij} | \eta'_i \eta'_j \rangle = \iint \psi_{\eta_i}(i) \psi_{\eta_j}(j) V_{ij} \psi_{\eta'_i}(i) \psi_{\eta'_j}(j) d\tau_i d\tau_j.$$

Operators in the second-quantization formulation are usually postulated as

$$\mathcal{W} \equiv \sum_{\eta_i \eta'_i} \langle \eta_i | W_i | \eta'_i \rangle b_{\eta_i}^+ b^{\eta'_i} \quad (24)$$

$$\mathcal{V} \equiv \frac{1}{2} \sum_{\substack{\eta_1 \eta_2 \\ \eta'_1 \eta'_2}} \langle \eta_1 \eta_2 | V_{12} | \eta'_1 \eta'_2 \rangle b_{\eta_2}^+ b_{\eta_1}^+ b^{\eta'_1} b^{\eta'_2} \quad (25)$$

( $\eta_i$  being the complete set of quantum numbers needed to specify the  $i^{\text{th}}$  single-particle state, i.e., states of the type  $p_i$  of the previous section). These postulates are considered valid if matrix

elements in the second-quantization scheme are identical with the corresponding ones in the usual Born-Heisenberg-Jordan scheme of matrix mechanics. Calculating the elements of  $\mathcal{W}$  between non-interacting states we have

$$\begin{aligned} & \langle p'_1 p'_2 \dots p'_N | \mathcal{W} | p_1 p_2 \dots p_N \rangle = \\ &= \sum_{\eta, \eta'} \langle \eta | W | \eta' \rangle \langle 0 | b^{p'_1} \dots b^{p'_2} b^{p'_1} (b_{\eta}^{\dagger}, b_{\eta'}^{\dagger}) b_{p_1}^{\dagger} b_{p_2}^{\dagger} \dots b_{p_N}^{\dagger} | 0 \rangle \\ &= \sum_{\eta, \eta'} \langle \eta | W | \eta' \rangle \langle 0 | b^{p'_1} \dots b^{p'_2} b^{p'_1} (\delta_{\eta}^{\eta'} - b_{\eta}^{\dagger} b_{\eta'}^{\dagger}) b_{p_1}^{\dagger} b_{p_2}^{\dagger} \dots b_{p_N}^{\dagger} | 0 \rangle \end{aligned}$$

having used the relations (12a) in the last step, so that

$$\begin{aligned} & \langle p'_1 p'_2 \dots p'_N | \mathcal{W} | p_1 p_2 \dots p_N \rangle = \\ & \sum_{\eta, \eta'} \langle \eta | W | \eta' \rangle \left\{ \delta_{\eta}^{\eta'} \delta_{p_1 p_2 \dots p_N}^{p'_1 p'_2 \dots p'_N} - \delta_{\eta, p_1 p_2 \dots p_N}^{\eta', p'_1 p'_2 \dots p'_N} \right\} \quad (26) \end{aligned}$$

which is indeed the usual result (see Condon & Shortley<sup>32)</sup> pp. 169-74 of matrix mechanics for the matrix elements of a single-body operator between two Slater determinant states  $\Phi'$  and  $\Phi$ . Carrying out a similar calculation for  $\mathcal{V}$  one obtains

$$\langle p'_1 p'_2 \dots p'_N | \mathcal{V} | p_1 p_2 \dots p_N \rangle =$$

$$\begin{aligned}
 &= \frac{1}{2} \sum_{\substack{\eta_1, \eta_2 \\ \eta'_1, \eta'_2}} \langle \eta_1, \eta_2 | V_{12} | \eta'_1, \eta'_2 \rangle \langle 0 | b^{\eta'_1} \dots b^{\eta'_2} b^{\eta_1} (b_{\eta_2}^+ b_{\eta_1}^+ b^{\eta'_1} b^{\eta'_2}) b_{\eta_1}^+ b_{\eta_2}^+ \dots b_{\eta_n}^+ | 0 \rangle \\
 &= \frac{1}{2} \sum_{\substack{\eta_1, \eta_2 \\ \eta'_1, \eta'_2}} \langle \eta_1, \eta_2 | V_{12} | \eta'_1, \eta'_2 \rangle \langle 0 | b^{\eta'_1} \dots b^{\eta'_2} b^{\eta_1} (\delta_{\eta_1}^{\eta'_1} \delta_{\eta_2}^{\eta'_2} - \delta_{\eta_1}^{\eta'_2} b_{\eta_2}^+ b_{\eta_1}^+ \\
 &\quad - \delta_{\eta_2}^{\eta'_1} \delta_{\eta_1}^{\eta'_2} + \delta_{\eta_2}^{\eta'_1} b_{\eta_2}^+ b_{\eta_1}^+ + \delta_{\eta_1}^{\eta'_2} b_{\eta_1}^+ b_{\eta_2}^+ - \delta_{\eta_2}^{\eta'_2} b_{\eta_1}^+ b_{\eta_1}^+ + b_{\eta_1}^+ b_{\eta_2}^+ b_{\eta_2}^+ b_{\eta_1}^+) \times \\
 &\quad \times b_{\eta_1}^+ b_{\eta_2}^+ \dots b_{\eta_n}^+ | 0 \rangle \quad (27)
 \end{aligned}$$

where the last step is arrived at by applying the anti-commutation relations (12 a,b,c) to the expression

$$b_{\eta_2}^+ b_{\eta_1}^+ b^{\eta'_1} b^{\eta'_2}$$

such as to push the creation operators to the right and the annihilation ones to the left. Noting in (27) that the terms

$$\delta_{\eta_1}^{\eta'_1} \delta_{\eta_2}^{\eta'_2} - \delta_{\eta_2}^{\eta'_1} \delta_{\eta_1}^{\eta'_2} \equiv \begin{vmatrix} \delta_{\eta_1}^{\eta'_1} & \delta_{\eta_2}^{\eta'_1} \\ \delta_{\eta_1}^{\eta'_2} & \delta_{\eta_2}^{\eta'_2} \end{vmatrix} \equiv \delta_{\eta_1, \eta_2}^{\eta'_1, \eta'_2}$$

and utilizing the scalar product formula (20) one arrives at the



final result that

$$\begin{aligned}
 & \langle p'_1 p'_2 \dots p'_N | \mathcal{V} | p_1 p_2 \dots p_N \rangle = \\
 & \frac{1}{2} \sum_{\substack{\eta_1, \eta_2 \\ \eta'_1, \eta'_2}} \langle \eta_1 \eta_2 | V_{12} | \eta'_1 \eta'_2 \rangle \left\{ \delta_{\eta_1 \eta'_1} \delta_{\eta_2 \eta'_2} \delta_{p_1 p'_1} \delta_{p_2 p'_2} \dots \delta_{p_N p'_N} \right. \\
 & - \delta_{\eta_1 \eta'_2} \delta_{\eta_2 \eta'_1} \delta_{p_1 p'_2} \delta_{p_2 p'_1} \dots \delta_{p_N p'_N} + \delta_{\eta_1 \eta'_1} \delta_{\eta_2 \eta'_2} \delta_{p_1 p'_2} \delta_{p_2 p'_1} \dots \delta_{p_N p'_N} \\
 & \left. - \delta_{\eta_1 \eta'_2} \delta_{\eta_2 \eta'_1} \delta_{p_1 p'_1} \delta_{p_2 p'_2} \dots \delta_{p_N p'_N} + \delta_{\eta_1 \eta'_2} \delta_{\eta_2 \eta'_1} \delta_{p_1 p'_2} \delta_{p_2 p'_1} \dots \delta_{p_N p'_N} \right\} \quad (28)
 \end{aligned}$$

whose properties are identical with those given in Condon & Shortley,<sup>32)</sup> pp. 169-74 the matrix elements of a two-body operator in the usual formulation. In conclusion, therefore, one has the required identities

$$\langle p'_1 p'_2 \dots p'_N | \mathcal{W} | p_1 p_2 \dots p_N \rangle = (\Phi', \sum_{i=1}^N W_i \Phi) \quad (29)$$

$$\langle p'_1 p'_2 \dots p'_N | \mathcal{V} | p_1 p_2 \dots p_N \rangle = (\Phi', \sum_{i < j=1}^N V_{ij} \Phi) \quad (30)$$

$\Phi$  and  $\Phi'$  being the Slater determinantal states (16) and (17), thus justifying postulates (24) and (25)

Operators in the second-quantization form are used extensively in current theoretical physics. When states  $|p_1 p_2 \dots p_N\rangle$  are given in terms of creation operators  $b_{p_i}^+$  as in (19), the usually difficulty and tedious problem of calculating matrix elements between superposed states of this type is reduced to the still tedious but intrinsically simpler task of anti-commuting  $b_p^+$ ,  $b^{p'}$  operators. In effect, however, even this is avoided as quicker and simpler techniques have been found for evaluating matrix elements. It may also be mentioned in passing that the second-quantization formulation is not necessarily restricted to the treatment of  $N$  fermions: a similar formulation can be derived replacing the relations (12 a,b,c) between fermi operators by commutation relations between boson operators for the treatment of such problems as phonon excitations in the vibrational nuclear model. Finally, we should anticipate the fact that contrary to second-quantization formulisms of the field-theory type, the total number of particles (nucleons, here) is always conserved as physical situations in low-energy nuclear physics demand.

Let us call the creation-annihilation pair  $b_{p_i}^+ b^{p'_i}$  by another name, for example

$$C_{p_i}^{p'_i} \equiv b_{p_i}^+ b^{p'_i} \quad (31)$$

where it is understood that this operator when acting on an arbitrary

state destroys a single-particle state given by the quantum numbers  $p_i'$  and creates one given by  $p_i$  explicitly,

$$C_{p_i}^{p_i'} |p_1 p_2 \dots p_i' \dots p_n\rangle = |p_1 p_2 \dots p_i \dots p_n\rangle \quad (31)$$

$$= 0 \text{ (if } p_i' \text{ is not present)}$$

We can find the commutation relations between the  $C_p^{p'}$  's by using the relations (12 a,b,c):

$$\begin{aligned} [C_p^{p'}, C_{\bar{p}}^{\bar{p}'}] &= [b_p^+ b^{p'}, b_{\bar{p}}^+ b^{\bar{p}'}] \\ &= b_p^+ [b^{p'}, b_{\bar{p}}^+ b^{\bar{p}'}] + [b_p^+, b_{\bar{p}}^+ b^{\bar{p}'}] b^{p'} \\ &= b_p^+ b_{\bar{p}}^+ [b^{p'}, b^{\bar{p}'}] + b_p^+ [b^{p'}, b_{\bar{p}}^+] b^{\bar{p}'} \\ &\quad + b_{\bar{p}}^+ [b_p^+, b^{\bar{p}'}] b^{p'} + [b_p^+, b_{\bar{p}}^+] b^{\bar{p}'} b^{p'} \\ &= 2 b_p^+ b_{\bar{p}}^+ b^{p'} b^{\bar{p}'} + b_p^+ b^{\bar{p}'} \delta_{\bar{p}}^{p'} - 2 b_p^+ b_{\bar{p}}^+ b^{p'} b^{\bar{p}'} \\ &\quad + 2 b_{\bar{p}}^+ b_p^+ b^{\bar{p}'} b^{p'} - b_{\bar{p}}^+ b^{p'} \delta_p^{\bar{p}'} - 2 b_{\bar{p}}^+ b_p^+ b^{\bar{p}'} b^{p'} \\ &= b_p^+ b^{\bar{p}'} \delta_{\bar{p}}^{p'} - b_{\bar{p}}^+ b^{p'} \delta_p^{\bar{p}'} \end{aligned}$$

$$\therefore [C_p^{p'}, C_{\bar{p}}^{\bar{p}'}] = C_p^{p'} \delta_{\bar{p}}^{\bar{p}'} - C_{\bar{p}}^{\bar{p}'} \delta_p^{p'}. \quad (32)$$

As Moshinsky realized<sup>33</sup>, this result is extremely important: the commutator of two  $C_p^{p'}$ 's is expressible as other  $C_p^{p'}$ 's. Exactly the same thing occurs with the angular momentum operators, namely,

$$[L_x, L_y] = i L_z \quad (\text{and cyclically})$$

are the generators for infinitesimal transformations (rotations) of the group  $R_3$ . (A better insight into the reason for calling  $L_x, L_y, L_z$  the  $R_3$  generators for infinitesimal rotations is given in Appendix C). Moshinsky's argument (see Appendix) concludes that relations (32) indicate that the set of operators  $\{C_p^{p'}\}$  forms the generators of a group of unitary transformations. The dimension of this group is the dimension of the space wherein the transformations take place, i.e., the possible values of the quantum number set  $\rho = \nu l m, \sigma \tau$ . Let the orbital sub-space  $\nu l m$  be  $r$ -dimensional, the spin-isospin sub-space is 4-dimensional since  $\sigma \tau = \frac{1}{2}\frac{1}{2}, \frac{1}{2}-\frac{1}{2}, -\frac{1}{2}\frac{1}{2}, -\frac{1}{2}-\frac{1}{2}$ . Thus the set of  $(4r)^2$  operators  $\{C_p^{p'}\}$  form the generators of a  $4r$ -dimensional unitary group  $U_{4r}$  whose Lie Algebra<sup>34</sup> is given by (32).

Now, the single-body operator  $\mathcal{W}$  of (24) is obviously the linear combination of  $U_{4r}$  group generators

$$\mathcal{W} = \sum_{p, p'} \langle p | \mathcal{W} | p' \rangle C_p^{p'} \quad (33)$$

with relatively simple coefficients as only one-particle states are involved. For the two-body operator  $\mathcal{V}$  of (25) we use (12 a,b,c) to convert

$$\begin{aligned}
 b_{p_2}^+ b_{p_1}^+ b_{p_1'} b_{p_2'} &= - b_{p_1}^+ b_{p_2}^+ b_{p_1'} b_{p_2'} \\
 &= - b_{p_1}^+ (\delta_{p_2}^{p_1'} - b_{p_2}^{p_1'} b_{p_1}^+) b_{p_2'} \\
 &= b_{p_1}^+ b_{p_1'} b_{p_2}^+ b_{p_2'} - \delta_{p_2}^{p_1'} b_{p_1}^+ b_{p_2'} \\
 &= C_{p_1}^{p_1'} C_{p_2}^{p_2'} - \delta_{p_2}^{p_1'} C_{p_1}^{p_2'}
 \end{aligned}$$

so that  $\mathcal{V}$  is the bilinear combination of  $U_4$  group generators

$$\mathcal{V} = \frac{1}{2} \sum_{\substack{p_1, p_2 \\ p_1', p_2'}} \langle p_1, p_2 | V_{12} | p_1', p_2' \rangle \{ C_{p_1}^{p_1'} C_{p_2}^{p_2'} - \delta_{p_2}^{p_1'} C_{p_1}^{p_2'} \} \quad (34)$$

with coefficients depending only on two-particle states.

A large class of problems involve one-and two-body operators which are independent of spin and isospin. Considering this restriction, and since

$$p_i \rightarrow \nu_i, l_i, m_i, \sigma_i, \tau_i \equiv \mu_i, S_i$$

where  $\mu_i$  refers to the configuration (orbital) space quantum numbers of the  $i^{th}$  particle and  $S_i$  to spin and isospin, the coefficients in (33) and (34) will be

$$\langle p_1 | W_1 | p_1' \rangle \equiv \langle \mu_1 s_1 | W_1 | \mu_1' s_1' \rangle = \langle \mu_1 | W_1 | \mu_1' \rangle \delta_{s_1 s_1'}$$

$$\begin{aligned} \langle p_1 p_2 | V_{12} | p_1' p_2' \rangle &= \langle \mu_1 s_1, \mu_2 s_2 | V_{12} | \mu_1' s_1', \mu_2' s_2' \rangle \\ &= \langle \mu_1, \mu_2 | V_{12} | \mu_1', \mu_2' \rangle \delta_{s_1 s_1'} \delta_{s_2 s_2'} \end{aligned}$$

because of the imposed independence of  $W_i$  and  $V_{ij}$  on  $S$ . Inserting these expressions in (33) and (34) one gets

$$W = \sum_{\mu, \mu'} \langle \mu | W_1 | \mu' \rangle \rho_{\mu}^{\mu'} \quad (35)$$

$$V = \frac{1}{2} \sum_{\substack{\mu, \mu_2 \\ \mu', \mu_2'}} \langle \mu, \mu_2 | V_{12} | \mu', \mu_2' \rangle \left\{ \rho_{\mu}^{\mu'} \rho_{\mu_2}^{\mu_2'} - \delta_{\mu_2 \mu_2'} \rho_{\mu}^{\mu'} \right\} \quad (36)$$

where the new operators  $\rho_{\mu}^{\mu'}$  are simply a contraction of the old ones over spin-isospin indices, that is

$$\rho_{\mu}^{\mu'} \equiv \sum_{ss'} \delta_s^{s'} C_{\rho}^{p'} = \sum_{ss'} \delta_s^{s'} C_{\mu s}^{\mu' s'}$$

$$\text{or } \rho_{\mu}^{\mu'} \equiv \sum_s C_{\mu s}^{\mu' s} \quad (\mu, \mu' = 1, 2, \dots, r) \quad (37)$$

Writing in (32)  $\mu_s$  for  $p$  and contracting over  $S$  we obtain the commutator algebra

$$[C_{\mu}^{\mu'}, C_{\bar{\mu}}^{\bar{\mu}'}] = C_{\mu}^{\bar{\mu}'} \delta_{\bar{\mu}}^{\mu'} - C_{\bar{\mu}}^{\mu'} \delta_{\mu}^{\bar{\mu}'} \quad (38)$$

which, by the argument upon (32) reveals that the  $r^2$  operators  $C_{\mu}^{\mu'}$  are the generators of an  $r$ -dimensional unitary group,  $U_r$ , of unitary transformations within the orbital sub-space of  $r$  dimensions.

From the analysis of this section, an analogy with the asymmetric top and its allowable energies of the Introduction is thus evident: the hamiltonian  $\mathcal{H}_T$  in (1) is quadratic in the  $R_3$  generators while here a nuclear hamiltonian  $\mathcal{H}$  with group symmetry greater than  $R_3$ , composed of both  $\mathcal{W}$  and  $\mathcal{V}$  types of interactions, is expressible as linear and bilinear combinations of the generators of a larger group, in general,  $U_{4r}$ . To evaluate the  $\|\mathcal{H}_T\|$  matrix of (3) in the  $|lm\rangle$  basis for an irreducible representation of  $R_3 > R_2$  only the matrix elements  $\langle lm' | L_1 | lm \rangle$  were needed. In our nuclear problem, to obtain the  $\|\mathcal{H}\|$  matrix we would need the matrix elements of the symmetry group generators. Although these are known for a unitary group of arbitrary dimension,<sup>75)</sup> other approaches will be necessary as solving the problem via the top analogy imposes great difficulties.

Our approach will be to construct a (truncated) basis which transforms irreducibly according to the various symmetry groups of a chosen portion of the model hamiltonian. To accomplish this, one begins with the problem of classifying the pertinent N-particle states with appropriate quantum numbers. However, a brief sketch of the treatment of this problem in  $R_3$  will be most useful.

#### 4.- Spherical Symmetry and the $R_3$ Group.

Consider first how one obtains in group theoretic language a basis for <sup>an</sup> irreducible representation of the group of three-dimensional rotations  $R_3$  and the labelling of these states. The generators of infinitesimal rotations around the x, y and z axes are respectively  $L_x$ ,  $L_y$  and  $L_z$ , where

$$\begin{aligned} L_x &= (\vec{r} \times \vec{p})_x = y p_z - z p_y = \frac{1}{i} \left( y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) \\ L_y &= (\vec{r} \times \vec{p})_y = z p_x - x p_z = \frac{1}{i} \left( z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right) \\ L_z &= (\vec{r} \times \vec{p})_z = x p_y - y p_x = \frac{1}{i} \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \end{aligned} \quad (39)$$

In spherical components one has the three generators

$$\begin{aligned} L_+ &\equiv L_x + i L_y \\ L_0 &\equiv L_z \\ L_- &\equiv L_x - i L_y \end{aligned} \quad (40)$$

which by  $[L_x, L_y] = i L_z$  (and cyclically) obey the Lie algebra

$$[L_0, L_{\pm}] = \pm L_{\pm}, \quad [L_+, L_-] = 2 L_0 \quad (41)$$

of which only  $L_0$  is hermitean since

$$(L_{\pm})^{\dagger} = L_{\mp} \quad \text{and} \quad L_0^{\dagger} = L_0,$$



$L_x, L_y, L_z$  being hermitean. Rotations are linear transformations and therefore the basis functions should be linearly independent homogeneous polynomials in the spherical space-components  $x_+, x_0$  and  $x_-$  of fixed degree, say  $h$ , which are solutions of the Euler equation

$$\vec{r} \cdot \nabla P(x_+, x_0, x_-) = h P(x_+, x_0, x_-). \quad (43)$$

These solutions however form a basis for a reducible representation. To decompose this set into irreducible sub-sets a further restriction on  $P$  must be imposed, namely

$$L_0 P = m P \quad (44)$$

using the hermitean operator  $L_0$  which commutes with the previous operator  $\vec{r} \cdot \nabla$  so that it can be employed simultaneously with  $\vec{r} \cdot \nabla$  to further characterize the polynomial by the integer

$m$ . The weight of the polynomial  $P$  is defined by  $m$ . Since  $L_+$  and  $L_-$  also commute with  $\vec{r} \cdot \nabla$  the polynomials  $P' = L_+ P$  and  $P'' = L_- P$  both satisfy (43). What weights do  $P'$  and  $P''$  have? Using the relations (41) one has

$$\begin{aligned} L_0 P' &= L_0 L_+ P = [L_0, L_+] P + L_+ L_0 P \\ &= L_+ P + m L_+ P \\ &= (m+1) P'. \end{aligned}$$

$$\begin{aligned}
 L_0 P'' &= L_0 L_- P = [L_0, L_-] P + L_- L_0 P \\
 &= -L_- P + m_- L_- P \\
 &= (m-1) P''
 \end{aligned}$$

The operator  $L_+$  thus raises the weight of an irreducible basis  $P$  by one unit and  $L_-$  lowers the weight by one unit.  $L_+$  and  $L_-$  are therefore called the raising and lowering operators of the group  $R_3$ , while  $L_0$  is the weight operator.

Among the solutions of (43) there are some polynomials of highest weight  $\mathbb{P}$  given by the three equations

$$\vec{r} \cdot \nabla \mathbb{P} = h \mathbb{P} \quad (45)$$

$$L_0 \mathbb{P} = l \mathbb{P} \quad (46)$$

$$\text{and} \quad L_+ \mathbb{P} = 0. \quad (47)$$

From (45) we take the general form

$$P(x_+, x_0, x_-) = \sum_{\alpha, \beta, \gamma} C_{\alpha\beta\gamma} x_+^\alpha x_-^\beta x_0^\gamma \quad (\alpha + \beta + \gamma = h),$$

in spherical components which, like (40), are  $x_\pm = x \pm iy$ ,  $x_0 = z$ . Applying the restriction (46) with  $L_0 = x_+ \partial/\partial x_+ - x_- \partial/\partial x_-$  gives the relation  $\beta = \alpha - l$  and (47) with  $L_+ = -(x_+ \partial/\partial x_0 + x_0 \partial/\partial x_-)$  gives a recurrence relation between the remaining coefficients so that

$$P(x_+, x_0, x_-) = A_0 x_+^l \sum_{\alpha=0}^{\frac{1}{2}(h-l)} \binom{\frac{h-l}{2}}{\alpha} x_0^{2(\frac{h-l}{2}-\alpha)} (-2x_+x_-)^\alpha$$

with  $A_0$  an arbitrary constant and  $\binom{a}{b}$  the binomial coefficients finally yields the solution to (45,46,47) as

$$P_h = A_0 x_+^l (x_0^2 - 2x_+x_-)^{\frac{h-l}{2}} \quad (48)$$

with  $h = l, l-2, l-4, \dots, 1, 0$  (49)

But the term  $(x_0^2 - 2x_+x_-)$  is simply  $r^2 = x^2 + y^2 + z^2$  so that one has

$$P_{hl} = A_0 x_+^l r^{h-l} \quad (h = l, l-2, l-4, \dots, 1, 0) \quad (51)$$

which (apart from an invariant  $r^{h-l}$  and a numerical constant) is the familiar solid spherical harmonic of maximum projection

$$P_{hl} \sim r^{h-l} Y_{ll}(\theta, \varphi)$$

$$Y_{ll}(r, \theta, \varphi) \equiv r^l Y_{ll}(\theta, \varphi). \quad (52a, b)$$

To summarize, the three equations

$$\vec{r} \cdot \nabla P = h P, \quad L_- P = l P, \quad L_+ P = 0$$

ensure solutions (51), or (52), which for the simplest case  $h=l$  (maximum representation) then  $P_l \sim Y_l(\theta, \varphi)$  is the maximum weight polynomial of the basis for the irreducible representation labeled by  $l$ . The full basis can be generated by successive application of the lowering operator  $L_-$ , giving us the rows  $m$  of the representation  $l$  thusly:

$$P_{lm} = \sqrt{\frac{(l+m)!}{(l-m)!(2l)!}} (L_-)^{l-m} P_l = r^l Y_{lm}(\theta, \varphi) \quad (53)$$

$$P_{lm} = Y_{lm}(r, \theta, \varphi)$$

where the radical in front of the lowering operation  $(L_-)^{l-m}$  is for normalization<sup>35)</sup> and  $m$  takes the  $(2l+1)$  allowed values

$$-l \leq m \leq l \quad (54)$$

which are unique for a given  $l$ -values, i.e., the set of numbers  $m$  labelling the rows is multiplicity free. The polynomial basis  $P_{lm}$  is clearly an eigenfunction of the commuting operators  $L^2$  and  $L_z$ ,

$$\begin{aligned} L^2 P_{lm} &= l(l+1) P_{lm} \\ L_z P_{lm} &= m P_{lm} \end{aligned} \quad (55a, b)$$

and the operator  $L^2$  is referred to as the Casimir operator<sup>36, 37, 38)</sup> / of  $R_3$  as 1) it is constructed from the  $R_3$  generators by

$$L^2 = L_x^2 + L_y^2 + L_z^2 = \sum_q (-1)^q L_q L_{-q} \quad (q = \pm 1, 0) \quad (56)$$

2) it commutes with them, namely  $[L^2, L_q] = 0$  and 3) it suffices to characterize each irreducible representation of  $R_3$  by the label  $l$ . The  $L_z$  operator generates rotations around the  $z$ -axis and is thus the single generator of the group  $R_2$  of axial rotations. Hence, the polynomial basis  $P_{lm}$  is said to be explicitly reduced with respect to the chain of groups

$$R_3 \supset \begin{pmatrix} R_2 & 0 \\ 0 & 1 \end{pmatrix} \quad (57)$$

the 1 in the lower-right hand corner coming from the fact that axial rotations here are about the  $z$ -axis.

Exactly analogous techniques have been developed<sup>39)</sup> for unitary group irreducible bases which are needed to construct wave functions for problems with symmetries larger than rotational.

#### 5.- Supermultiplet Classification of States by Unitary Groups. Permutational Symmetry.

The group of unitary transformations within the total orbital-spin-isospin space of  $4r$  dimensions is  $U_{4r}$  whose  $(4r)^2$  generators are  $C_i^{p'} \equiv b_p^\dagger b^{p'}$ ,  $p$  designating the components, which obey the Lie commutator algebra (32). Our transformations are linear, so the polynomial basis must be a set of linearly independent homogeneous polynomials  $P$  of degree  $N$  in the  $b_p^\dagger$ . The set more-

over corresponds to the completely antisymmetric representation  $[1^N 0^{4r-N}]$  of  $U_{4r}$ . The scalar product between two such polynomials is defined as

$$(P_1, P_2) \equiv \langle 0 | P_1^\dagger P_2 | 0 \rangle \quad (58)$$

where in  $P_1^\dagger$ , all  $b_p^\dagger$  are replaced by  $b_p$ , and thus evaluation of (58) depends only on the properties (12 a,b,c) and (11). The creation and annihilation operators themselves transform as

$$U_{4r} : \quad \begin{aligned} b_{p'}^\dagger &= \sum_p U_{p'}^p b_p^\dagger \\ b^{p'} &= \sum_p (U_{p'}^p)^* b^p \end{aligned} \quad (p = 1, 2, \dots, 4r) \quad (59)$$

the  $U_{p'}^p$  being elements of a unitary matrix.

As already seen in (37), contraction of  $C_p^{p'} \equiv C_{\mu s}^{\mu' s'}$  over the index  $s$  gives an operator set  $\{E_\mu^{\mu'}\}$  with Lie algebra (38) and therefore constitute the generators of a group  $U_r$  of unitary transformations in the orbital space of  $r$  dimensions, i.e.,

$$U_r : \quad \begin{aligned} b_{\mu' s}^\dagger &= \sum_\mu U_{\mu'}^\mu b_{\mu s}^\dagger \\ b^{\mu' s} &= \sum_\mu (U_{\mu'}^\mu)^* b^{\mu s} \end{aligned} \quad \begin{aligned} (\mu = 1, 2, \dots, r) \\ (s = 1, 2, 3, 4) \end{aligned} \quad (60)$$

Moreover, we can form the  $4^2=16$  operators

$$C_s^{s'} \equiv \sum_{\mu} \delta_{\mu}^{\mu'} C_{\mu s}^{\mu' s'} = \sum_{\mu} C_{\mu s}^{\mu s'} \quad (s, s' = 1, 2, 3, 4) \quad (61)$$

by contraction over the index  $\mu$ , and see immediately from (32), contracted in the same manner, that

$$[C_s^{s'}, C_{s''}^{s'''}] = C_s^{s'''} \delta_{s''}^s - C_{s''}^{s'} \delta_s^{s'''} \quad (62)$$

is a Lie algebra identical with (38) and (32) so that the operator set  $\{C_s^{s'}\}$  form the generators of a group  $U_4$  of transformations

$$U_4 : \quad \begin{aligned} b_{\mu s'}^{\dagger} &= \sum_s U_{s'}^s b_{\mu s}^{\dagger} \\ b^{\mu s'} &= \sum_s (U_{s'}^s)^* b^{\mu s}. \end{aligned}$$

The generators of  $\mathcal{U}_V$  commute with those of  $U_4$ :

$$\begin{aligned} [C_{\mu}^{\mu'}, C_s^{s'}] &= \left[ \sum_{\bar{s}} C_{\mu \bar{s}}^{\mu' \bar{s}}, \sum_{\bar{\mu}} C_{\bar{\mu} s}^{\bar{\mu} s'} \right] \\ &= \sum_{\bar{s}, \bar{\mu}} [C_{\mu \bar{s}}^{\mu' \bar{s}}, C_{\bar{\mu} s}^{\bar{\mu} s'}] \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\bar{s}, \bar{\mu}} \left\{ C_{\mu \bar{s}}^{\bar{\mu} s'} \delta_{\bar{\mu}}^{\mu'} \delta_{\bar{s}}^{\bar{s}'} - C_{\bar{\mu} s}^{\mu' s'} \delta_{\mu}^{\bar{\mu}} \delta_{\bar{s}}^{s'} \right\} \\
 &= \sum_{\bar{s}, \bar{\mu}} \left\{ C_{\mu s}^{\mu' s'} - C_{\mu s}^{\mu' s'} \right\} \\
 &= 0. \tag{64}
 \end{aligned}$$

recalling that  $\delta_{\rho}^{\rho'} \equiv \delta_{\mu s}^{\mu' s'} = \delta_{\mu}^{\mu'} \delta_s^{s'}$ . Hence, the group  $U_{4r}$  of matrices  $\|U_{\mu s}^{\mu' s'}\|$  contains the product sub-group  $U_r \times U_4$  of matrices  $\|U_{\mu}^{\mu'} U_s^{s'}\|$  and this contains separately the sub-groups  $U_r$  with  $\|U_{\mu}^{\mu'} \delta_s^{s'}\|$  and  $U_4$  with  $\|\delta_{\mu}^{\mu'} U_s^{s'}\|$ . In shorthand:

$$\begin{aligned}
 &\underset{\substack{\uparrow \\ \text{(orbital-spin-isospin)}}}{U_{4r}} > U_r \times U_4 \left\{ \begin{array}{l} > U_r \leftarrow \text{(orbital)} \\ > U_4 \leftarrow \text{(spin-isospin)} \end{array} \right. \tag{65}
 \end{aligned}$$

the names in parentheses referring to the single-particle transformation space.

Let us now define the irreducible representations connected with this chain of groups. The  $r$  operators  $\ell_{\mu}^{\mu}$  and 4 operators  $C_s^s$  are hermitean since



$$\begin{aligned} (b_{\mu}^{\mu})^{\dagger} &\equiv \sum_s (b_{\mu s}^{\dagger} b^{\mu s})^{\dagger} = \sum_s b_{\mu s}^{\dagger} b^{\mu s} = b_{\mu}^{\mu} \\ (C_s^s)^{\dagger} &\equiv \sum_{\mu} (b_{\mu s}^{\dagger} b^{\mu s})^{\dagger} = \sum_{\mu} b_{\mu s}^{\dagger} b^{\mu s} = C_s^s \end{aligned} \quad (66)$$

while  $b_{\mu}^{\mu'}$  ( $\mu \neq \mu'$ ) and  $C_s^{s'}$  ( $s \neq s'$ ) have the hermiticity properties

$$(b_{\mu}^{\mu'})^{\dagger} = b_{\mu'}^{\mu} ; (C_s^{s'})^{\dagger} = C_{s'}^s.$$

In analogy with the  $R_3$  hermitean operator  $L_0$  one can define, as in (44), in  $\mathcal{U}_r$  and simultaneously in  $\mathcal{U}_4$ , the weight of the set of homogeneous polynomials  $P$  of degree  $N$  by

$$b_{\mu}^{\mu} P = w_{\mu} P \quad (\mu=1,2,\dots,r) \quad (67)$$

$$C_s^s P = \omega_s P \quad (s=1,2,3,4) \quad (68)$$

since the operator sets commute mutually, so that the weight of  $P$  in  $\mathcal{U}_r$  is the set of numbers  $[w_1, w_2, \dots, w_r]$ , and in  $\mathcal{U}_4$  it is  $[\omega_1, \omega_2, \omega_3, \omega_4]$ . From (67), the integer  $w_{\mu}$  is the degree of  $P$  in  $b_{\mu s}$  with respect to the index  $\mu$  while, from (68), the integer  $\omega_s$  gives the degree with respect to  $s$ . Example:

$$\underline{N=5}: P = b_{11}^{\dagger} b_{12}^{\dagger} b_{22}^{\dagger} b_{32}^{\dagger} b_{44}^{\dagger}$$

has  $\mathcal{U}_r$  weight  $[w_1, w_2, \dots, w_r] = [2, 1, 1, 0, 0, \dots, 0]$  and  $\mathcal{U}_4$  weight

$[w_1, w_2, w_3, w_4] = [1301]$  . The sum of individual degrees

$$\sum_{\mu=1}^r w_{\mu} = \sum_{s=1}^4 w_s = N$$

is the total number of particles, which in turn is the eigenvalue of the number operator

$$N = \sum_{\mu=1}^r b_{\mu}^{\mu} = \sum_{s=1}^4 C_s^s \quad (71)$$

$$N P = N P$$

A polynomial  $P$  of weight  $[w_1, w_2, \dots, w_r]$  in  $\mathcal{U}_r$  is of higher weight than  $P'$  of weight  $[w'_1, w'_2, \dots, w'_r]$  in  $\mathcal{U}_r$  if in the set of differences

$$[w_1 - w'_1, w_2 - w'_2, \dots, w_r - w'_r] \quad (72)$$

the first non-zero number is positive. Likewise, the same criterion for higher and lower weights in  $\mathcal{U}_4$  applies.

Consider now the remaining operators of  $\mathcal{U}_r$ , i.e.,  $b_{\mu}^{\mu'} (\mu \neq \mu')$  and of  $\mathcal{U}_4$ , i.e.,  $C_s^{s'} (s \neq s')$ . Take the two polynomial classes  $P'$  and  $P''$  as

$$\begin{aligned} P' &= b_{\mu}^{\mu'} P \\ P'' &= b_{\mu'}^{\mu} P \end{aligned} \quad (\mu < \mu')$$

Using (11), (12 a,b,c) and (67), one can find their weights:

$$e_{\mu}^{\mu} P' = e_{\mu}^{\mu} e_{\mu}^{\mu'} P = [e_{\mu}^{\mu}, e_{\mu}^{\mu'}] P + e_{\mu}^{\mu'} e_{\mu}^{\mu} P$$

$$= e_{\mu}^{\mu'} P + w_{\mu} e_{\mu}^{\mu'} P = (w_{\mu} + 1) P'$$

$$e_{\mu'}^{\mu'} P' = e_{\mu'}^{\mu'} e_{\mu}^{\mu} P = [e_{\mu'}^{\mu'}, e_{\mu}^{\mu}] P + e_{\mu}^{\mu} e_{\mu'}^{\mu'} P$$

$$= -e_{\mu}^{\mu} P + w_{\mu'} e_{\mu}^{\mu} P = (w_{\mu'} - 1) P'$$

$$e_{\mu}^{\mu} P'' = (w_{\mu} - 1) P''$$

$$e_{\mu'}^{\mu'} P'' = (w_{\mu'} + 1) P''$$

so that  $P'$  has weight  $[w_1, \dots, w_{\mu}+1, \dots, w_{\mu'}-1, \dots, w_r]$  and  $P''$  weighs  $[w_1, \dots, w_{\mu}-1, \dots, w_{\mu'}+1, \dots, w_r]$  which are respectively higher and lower than the weight  $[w_1, w_2, \dots, w_{\mu}, \dots, w_{\mu'}, \dots, w_r]$  of  $P$ . Thus in analogy to  $L_+$  and  $L_-$  of  $R_3$ ,  $e_{\mu}^{\mu'} (\mu < \mu')$  are the raising and  $e_{\mu}^{\mu'} (\mu > \mu')$  the lowering operators of the group  $U_r$ . In the same manner, one can see that  $C_s^{s'} (s < s')$  and  $C_s^{s'} (s > s')$  are the raising and lowering operators of  $U_q$ .  
Thus

WEIGHT OPERATORS

RAISING

$U_r:$

$$\{e_{\mu}^{\mu'}\} \equiv \begin{Bmatrix} e_1^1 & e_1^2 & e_1^3 & \dots & e_1^r \\ e_2^1 & e_2^2 & e_2^3 & \dots & \\ e_3^1 & e_3^2 & e_3^3 & \dots & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e_r^1 & e_r^2 & \dots & \dots & e_r^r \end{Bmatrix} \quad (72)$$

LOWERING

and a similar array for  $\{C_s^{s'}\}$  of  $U_4$  clearly show the three categories of raising, lowering and weight operators present in unitary groups which is the generalization for these groups of the similar result obtained previously for  $R_3$ .

Among the set of linearly independent polynomials  $P$  which satisfy equations (67) and (68) thus being characterized by both  $[\omega_1, \omega_2, \dots, \omega_r]$  and  $[\omega_1, \omega_2, \omega_3, \omega_4]$ , there are some, say  $P$ , of highest weight in both  $U_r$  and  $U_4$  given by

$$e_{\mu}^{\mu} P = h_{\mu} P, \quad e_{\mu}^{\mu'} P = 0 \quad \left( \begin{matrix} \mu, \mu' = 1, 2, \dots, r \\ \mu < \mu' \end{matrix} \right) \quad (73a, b)$$

and simultaneously by

$$C_s^s P = v_s P, \quad C_s^{s'} P = 0 \quad \left( \begin{array}{l} s, s' = 1, 2, 3, 4 \\ s < s' \end{array} \right) \quad (74a, b)$$

the highest weight polynomial set  $\underline{P}$  now being characterized by  $[h_1, h_2, \dots, h_r] \times \{v_1, v_2, v_3, v_4\}$ , which is the representation label of the direct product group  $\mathcal{U}_r \times \mathcal{U}_4$ . The set of equations (73) and (74) for  $\mathcal{U}_r \times \mathcal{U}_4$  is analogous to the set (46) and (47) for  $R_3$ : they select the maximum weight functions  $\underline{P}$  out of a larger set  $P$ . Should one further insist on an analogy here of the Euler equation (45) treated in the discussion of  $R_3$ , we can cite

$$\sum_P C_P P = N P$$

which like (45) gives the total degree of  $\underline{P}$ .

We recall from (53) of our discussion of the rotation group that the complete base  $P_{lm}$ , corresponding to irreducible representation  $l$  of  $R_3$ , where the row index  $m$  is given by  $-l \leq m \leq l$ , can be generated from the maximum weight and  $h=l$  polynomial  $P_l \sim \chi_l^l$  of (51) by successive applications of the  $R_3$  lowering generator  $L_-$ .

In the case of  $\mathcal{U}_r \times \mathcal{U}_4$ , it is possible also to generate the full basis ---- labeled by the appropriate rows to be discussed further on ---- via application of operators which are functions of the lowering generators of this group. The representation  $[h_1, h_2, \dots, h_r] \times \{v_1, v_2, v_3, v_4\}$  of  $\mathcal{U}_r \times \mathcal{U}_4$  is thus definitely irreducible. Moreover, this irreducible representation is unique (See Appendix B - ).

The direct product group  $\mathcal{U}_r \times \mathcal{U}_4$  contains  $\mathcal{U}_r$  and  $\mathcal{U}_4$

as subgroups. The highest weight term of an irreducible basis for  $U_r$  and  $U_4$ , labelled respectively by  $[h_1, h_2, \dots, h_r]$  and  $\{v_1, v_2, v_3, v_4\}$ , obey restrictions analogous to (49) for the irreducible representations of  $R_3$ . We recall that (49) followed simply from the fact that  $P$  was a homogeneous polynomial in  $\chi_+$ ,  $\chi_0$  and  $\chi_-$ . In the present case, if one considers the non-negative scalar product

$$\begin{aligned}
 (C_{\mu}^{\mu'} P, C_{\mu}^{\mu'} P) &= \langle 0 | P^{\dagger} C_{\mu'}^{\mu} C_{\mu}^{\mu'} P | 0 \rangle \quad (\mu > \mu') \\
 &= \langle 0 | P^{\dagger} [C_{\mu'}^{\mu}, C_{\mu}^{\mu'}] P | 0 \rangle \\
 &= \langle 0 | P^{\dagger} (C_{\mu'}^{\mu'} - C_{\mu}^{\mu}) P | 0 \rangle \\
 &= (h_{\mu'} - h_{\mu}) (P, P) \geq 0
 \end{aligned}
 \tag{75}$$

$$\therefore \underline{h_{\mu} \geq h_{\mu'}} \quad \text{for } (\mu < \mu')$$

where in step 2 we used (73b), in 3 (73a) and in the last step the fact  $(P, P) \geq 0$ . Similar results are obtained from the scalar product  $(C_s^{s'} P, C_s^{s'} P)$  for  $s > s'$ . Furthermore, from (71) one obtains  $\sum_{\mu=1}^r h_{\mu} = \sum_{s=1}^4 v_s = N$ . In conclusion, we have the following conditions:

$$U_4: \quad \begin{aligned} V_1 &\geq V_2 \geq V_3 \geq V_4 \geq 0 \\ V_1 + V_2 + V_3 + V_4 &= N \end{aligned} \quad (77)$$

The unique highest weight function of an arbitrary orbital permutation symmetry and the corresponding (conjugate) spin-isospin permutation symmetry can be constructed at once. For instance, take the 13 particle orbital symmetry given by the partition

$$u_r: [h_1, h_2, \dots, h_6] = [4, 3, 2, 2, 1, 1] =$$

where obviously  $4+3+2+2+1+1 = 13$ . A polynomial function  $P(b_{\mu s}^+)$  with this  $U_r$  irreducible representation and of maximum weight in  $U_r$  is simply

$$\begin{aligned}
 \mathbb{P}(b_{\mu s}^+) = & \begin{matrix} + & + & + & + \\ b_{11} & b_{12} & b_{13} & b_{14} \\ \times & b_{21}^+ & b_{22}^+ & b_{23}^+ \\ & \times & b_{31}^+ & b_{32}^+ \\ & & \times & b_{41}^+ & b_{42}^+ \\ & & & \times & b_{51}^+ \\ & & & & \times & b_{61}^+ \end{matrix} \quad (79)
 \end{aligned}$$

which is clearly totally anti-symmetric with respect to interchange of any pair of orbital-spin-isospin states. Note that the arrangement of  $b^+$ 's purposely follows the Young diagram (78). The degrees  $h_\mu$  of  $\mathbb{P}$  with respect to the components  $\mu$  in  $b_{\mu s}^+$  is  $h_1 = 4$  so that four states with  $\mu = 1$  are constructed;  $h_2 = 3$  and three states with  $\mu = 2$  are constructed, and so forth. The second index of each  $b^+$  operator ---- denoting the spin-isospin state ---- is placed in numerical order in each row. (If two of these indices coincide the whole expression vanishes as from (12b) any  $b_{\mu s}^+ b_{\mu s}^+ = 0$ , i.e., the Pauli principle). Naturally therefore as  $1 \leq s \leq 4$ , there are  $\leq$  four blocks in any given row. By the same taken, there are  $\leq r$  (=number of single nucleon orbital states available) rows in the whole expression.

Mental application of  $c_\mu^\mu$  ( $\mu = 1, 2, \dots, 6$ ) shows that



indeed  $[h_1 h_2 h_3 h_4 h_5 h_6] = [4 3 2 2 1 1]$ . The effect of any  $b_{\mu}^{\mu'}$  ( $\mu < \mu'$ ) is zero as in some column a factor  $b_{\mu s}^+ b_{\mu s}^+ = 0$  would then occur, thus (78) is of maximum weight in  $U_r$ .

Again, mental application of  $C_s^{s'}$  ( $s = 1, 2, 3, 4$ ) shows that  $[V_1 V_2 V_3 V_4]$ , the degrees of  $\mathbb{P}$  in  $S=1, 2, 3, 4$ , is

$[V_1 V_2 V_3 V_4] = [6 4 2 1]$ , which satisfy (77), and furthermore is the conjugate representation of  $[h_1 h_2 h_3 h_4 h_5 h_6]$  as

$$[h_1 h_2 h_3 h_4 h_5 h_6] = [\widetilde{4 3 2 2 1 1}] = [6 4 2 1] = [V_1 V_2 V_3 V_4] \quad (80)$$

as can be seen by reflecting the diagram (78) about its principal diagonal. (This was to be expected since the irreducible representation  $[h_1 h_2 \dots h_r] \times \{V_1 V_2 V_3 V_4\}$  of  $U_r \times U_4$  is contained in the irreducible representation  $[1^n 0^{4r-n}]$  of  $U_{4r}$ ). Moreover, (79) satisfies  $C_s^{s'} \mathbb{P} = 0$ , for  $s < s'$ , since the effect of any of these  $C_s^{s'}$  would create pairs of the form  $b_{\mu s}^+ b_{\mu s}^+ = 0$ .

The state (79) is the analogue for the  $U_r \times U_4$  group of (51). Extremely powerful and simple is the technique of constructing any  $\mathbb{P}^{[h_1 h_2 \dots h_r] \times \{V_1 V_2 V_3 V_4\}}$  of maximum weight. Lowering operations as in (53) for obtaining the complete bases  $P_{\ell m}$ , irreducible under  $R_3 > R_2$ , with  $-\ell \leq m \leq \ell$  and of dimension  $(2\ell + 1)$ , also have their analogue in unitary group theory.

The polynomial set  $P_{\ell m}$  transforms irreducibly under  $R_3$  according to the well-known "rotation matrices"<sup>40)</sup>

$$P_{\ell m}'(x, y, z) = \sum_{m'} D_{mm'}^{(\ell)} P_{\ell m'}(x, y, z). \quad (81)$$

The row of the irreducible representation  $D^{(\ell)}$  of dimension

$(2l+1)$  by  $(2l+1)$  is given by  $m$  which, apart from being the polynomial weight in  $R_2$ , is also the irreducible representation of  $R_2$  which is contained in  $R_3$  as a subgroup, i.e.,  $P_{lm}$  transforms irreducibly under the chain of group transformations designated as

$$R_3 \supset \begin{pmatrix} R_2 & 0 \\ 0 & 1 \end{pmatrix} \quad (82)$$

Similarly, the rows (and columns) that would specify the full basis of a polynomial set  $P^{[h_1 h_2 \dots h_r]} (b_{\mu s}^+)$  transforming irreducibly under  $U_r$  could come from sub-groups of  $U_r$ . Not any chain of sub-groups, however, would provide a complete classification of the polynomial set. For example, the chains

$$U_6 \supset U_3, \quad U_4 \supset U_2 \times U_2, \quad U_6 \supset R_6 \quad (83)$$

among other physically important cases<sup>37, 41)</sup> are known as "non-simply reducible" because for a given irreducible representation of the large group there may be repeated sub-group irreducible representations thus requiring additional labels<sup>39)</sup> (quantum numbers, physical or not) to distinguish these multiplicities. However, the so-called "canonical chain"<sup>39)</sup>

$$U_r \supset \begin{pmatrix} U_{r-1} & 0 \\ 0 & 1 \end{pmatrix} \supset \begin{pmatrix} U_{r-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \supset \dots \supset \begin{pmatrix} U_1 & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & \ddots \\ & & & & 1 \end{pmatrix} \quad (84)$$

unequivocally specifies the rows of a  $U_r$  irreducible representation.

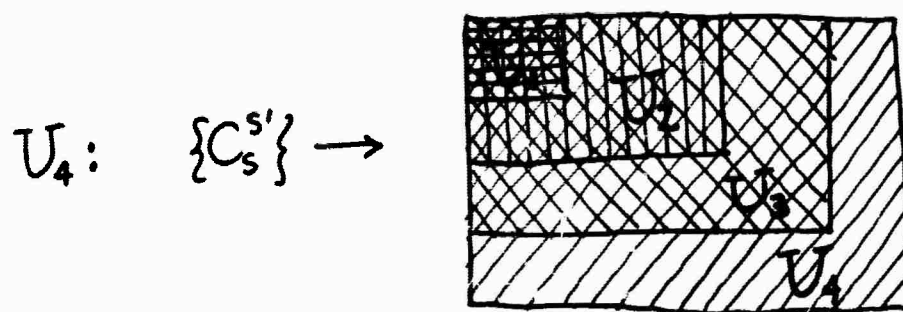
The "canonical chain" for the spin-isospin group  $U_4$  would hence be

$$U_4 \supset \begin{pmatrix} U_3 & 0 \\ 0 & 1 \end{pmatrix} \supset \begin{pmatrix} U_2 & 0 & 0 \\ 0 & 1 & \\ 0 & & 1 \end{pmatrix} \supset \begin{pmatrix} U_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (85)$$

and as this is smaller than  $U_r$  in general, let us illustrate the sub-group generators. The  $4^2$  generators of  $U_4$  are

$$U_4: \{C_s^{s'}\} \equiv \left\{ \begin{array}{c|c|c|c} \overbrace{C_1^1} & C_1^2 & C_1^3 & C_1^4 \\ \hline C_2^1 & C_2^2 & C_2^3 & C_2^4 \\ \hline C_3^1 & C_3^2 & C_3^3 & C_3^4 \\ \hline C_4^1 & C_4^2 & C_4^3 & C_4^4 \end{array} \right\} \quad (86)$$

with Lie algebra  $[C_s^{s'}, C_s^{s''}] = C_s^{s''} \delta_s^{s'} - C_s^{s'} \delta_s^{s''}$  ( $s, s' = 1, 2, 3, 4$ ), namely the commutator of any two generators never gives other than  $U_4$  generators. The  $U_3$  generators can be chosen as the  $3^2$  operators in the upper-left hand corner above the solid lines. They satisfy the same Lie algebra as  $U_4$ , but with  $s, s' = 1, 2, 3$  ---- as can be quickly verified. The  $U_2$  generators  $C_1', C_2^2, C_2', C_1^2$  also obey the same algebra, but with  $s, s' = 1, 2$ . Finally,  $C_1'$  is the sole generator of  $U_1$  and the commutator  $[C_1', C_1'] = 0$  also gives nothing outside  $U_1$ . An illustrative diagram showing the group chain (85) insofar as the  $4^2$  generators  $C_s^{s'} (s, s' = 1, 2, 3, 4)$  are concerned is:



Generalizing,  $U_r$  will from (84) have  $(r-1)$  sub-groups  $U_p$  ( $p=1, 2, \dots, r-1$ ) each with  $p^2$  generators  $C_\mu^{\mu'} (\mu, \mu' = 1, 2, \dots, p)$ . Moreover, the equations

$$C_\mu^{\mu} P = h_{\mu} P \quad C_\mu^{\mu'} P = 0 \quad (87)$$

$$(\mu = 1, 2, \dots, p) \quad (\mu < \mu') \quad (p = 1, 2, \dots, r-1)$$

define the irreducible representation label  $[h_{1p} h_{2p} \dots h_{pp}]$  of the sub-group  $U_p \subset U_r$ . Likewise,

$$C_s^s P = v_{s_g} P \quad C_s^{s'} P = 0 \quad (88)$$

$$(s=1,2,\dots,g) \quad (g=1,2,3) \quad (s < s')$$

define the irreducible representation  $\{v_{1g} v_{2g} \dots v_{gg}\}$  of the subgroup  $U_g \subset U_4$ .

Hence, the generic state forming a complete basis for irreducible representations of the groups in the chain

$$U_{4r} \supset U_r \times U_4 \supset \begin{cases} U_r \supset \begin{pmatrix} U_{r-1} & 0 \\ 0 & 1 \end{pmatrix} \supset \dots \supset \begin{pmatrix} U_1 & 0 \\ 0 & 1 \dots 1 \end{pmatrix} \end{cases} \quad (89a)$$

$$U_{4r} \supset U_r \times U_4 \supset \begin{cases} U_4 \supset \begin{pmatrix} U_3 & 0 \\ 0 & 1 \end{pmatrix} \supset \begin{pmatrix} U_2 & 0 \\ 0 & 1 \end{pmatrix} \supset \begin{pmatrix} U_1 & 0 \\ 0 & 1 \end{pmatrix} \end{cases} \quad (89b)$$

could be designated with a notation used by Gel'fand & Zetlin<sup>42)</sup> as:

$$P^{[h_{kp}] \times \{v_{fg}\}}(b_{\mu s}^+) \equiv |h_{kp}; v_{fg}\rangle \equiv$$

$$\begin{array}{ccc}
 h_{1r} h_{2r} \dots h_{rr} & & \\
 h_{1r-1} h_{2r-1} \dots h_{r-1r-1} & & \\
 \dots & & \\
 h_{12} h_{22} & & \\
 h_{11} & & 
 \end{array}
 \begin{array}{c}
 \cdot \\
 ; \\
 \cdot
 \end{array}
 \begin{array}{cccc}
 V_{14} & V_{24} & V_{34} & V_{44} \\
 V_{12} & V_{22} & V_{32} & \\
 V_{13} & V_{23} & & \\
 V_{14} & & & 
 \end{array}
 \begin{array}{c}
 \diagup \\
 \diagdown
 \end{array}
 (90)$$

with  $p \leq k = 1, 2, \dots, r$  and  $q \leq f = 1, 2, 3, 4$ ,

which, for the chain (89), is the analogue of (53) for the simple chain (57). In the latter, one had the restriction that

$-l \leq m \leq l$ . An analogous restriction holds in (89), for both

$U_r$  and  $U_4$  chains: Let  $\pi_N$  be the symmetric (or permutation) group of  $N$  objects. As  $\pi_N > \pi_{N-1}$ , the former having irreducible representations (Young patterns)  $[f_1 f_2 \dots f_N]$  and the latter  $[f'_1 f'_2 \dots f'_{N-1}]$ , then

$$f_1 \geq f'_1 \geq f_2 \geq f'_2 \geq \dots \geq f'_{N-1} \geq f_N \geq 0 \quad (91)$$

(see Weyl, ref. 43). Because of the intimate relation between

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the  $\pi_N$  and  $\mathcal{U}_r$  groups<sup>44)</sup>, namely, that they possess the same irreducible representation labels, one deduces from (91) the following inequalities for  $\mathcal{U}_r$  and its canonical chain subgroups:

$$\begin{aligned} h_{1r} &\geq h_{1r-1} \geq h_{2r} \geq h_{2r-1} \geq \dots \geq h_{r-1r-1} \geq h_{rr} \geq 0 \\ h_{1r-1} &\geq h_{1r-2} \geq \dots \geq h_{r-2r-2} \geq h_{r-1r-1} \geq 0 \\ &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ h_{12} &\geq h_{11} \geq h_{22} \geq 0 \end{aligned}$$

or, to use the same arrangement as in the generic state (90),

$$\begin{array}{ccccccc} h_{1r} & & h_{2r} & & h_{3r} & \cdot & \cdot & \cdot & h_{rr} \geq 0 \\ \Rightarrow & \nearrow & \Rightarrow & \nearrow & \Rightarrow & \nearrow & & \nearrow & \\ & h_{1r-1} & & h_{2r-1} & & h_{3r-1} & \cdot & \cdot & h_{r-1r-1} \geq 0 \\ & \Rightarrow & \nearrow & \Rightarrow & \nearrow & \Rightarrow & & \nearrow & \\ & & & & & & & & \geq 0 \\ & & & & \Rightarrow & h_{12} & \nearrow & h_{22} & \geq 0 \\ & & & & & \Rightarrow & h_{11} & \nearrow & \end{array} \quad (92)$$



For  $U_4$  and its sub-groups, the same reasoning in (91) gives the requirements

$$\begin{array}{ccccccc}
 V_{14} & & V_{24} & & V_{34} & & V_{44} \geq 0 \\
 & \Rightarrow & & \Rightarrow & & \Rightarrow & \\
 & & V_{13} & & V_{23} & & V_{33} \geq 0 \\
 & & & \Rightarrow & & \Rightarrow & \\
 & & & & V_{12} & & V_{22} \geq 0 \\
 & & & & & \Rightarrow & \\
 & & & & & & V_{11}
 \end{array}
 \quad (93)$$

The polynomial  $\mathbb{P}^{[h_1 h_2 \dots h_r] \times \{V_1 V_2 V_3 V_4\}}$  of highest weight in both  $U_r$  and  $U_4$  must satisfy both (73) and (74). Now, the conditions (73b) and (74b), i.e.,

$$C_\mu^{\mu'} P = 0 \quad (\mu < \mu'), \quad C_s^{s'} P = 0 \quad (s < s')$$

ensure that  $\mathbb{P}$  is automatically of highest weight in all the canonical chain subgroups of  $U_r$  and  $U_4$ .

Concretely, if the representation  $[h_{kr}]$  ( $k=1,2,\dots,r$ ) of  $U_r$  is of highest weight, the sub-group representations  $[h_{kp}]$  ( $k \leq p=1,2,\dots,r-1$ ) take the maximum value compatible with (92) and if the  $U_4$  representation  $\{V_{f4}\}$  ( $f=1,2,3,4$ ) is of highest weight, its sub-group representations  $\{V_{fg}\}$  ( $f \leq g=1,2,3$ ) acquire the ~~maximum~~ values allowed by (93). This result will be very significant in future developments, but for the moment we note that in the explicit Gelfand-Zetlin notation the solution of (73) and (74) (a)

can consequently be symbolized in the simple form

$$P \equiv P^{[h_{1r} h_{2r} \dots h_{rr}] \{v_{14} v_{24} v_{34} v_{44}\}} = \quad (94)$$

$$\left| \begin{array}{cccc} h_{1r} & h_{2r} & h_{3r} & \dots & h_{rr} \\ & h_{1r} & h_{2r} & h_{3r} & \dots & h_{r-1r} \\ & & . & . & . & \\ & & & h_{1r} & h_{2r} \\ & & & & h_{1r} \end{array} \right. ; \begin{array}{cccc} v_{14} & v_{24} & v_{34} & v_{44} \\ & v_{14} & v_{24} & v_{34} \\ & & v_{14} & v_{24} \\ & & & v_{14} \end{array}$$

where the h's along any diagonal parallel to the lowest one are equal among themselves. The same holds for the v's.

Equations (67) and (68) define the weights in  $U_r$  and  $U_4$  of a given polynomial state such as (90) ---- weights being nothing more than the eigenvalues of  $C_\mu^\mu$  ( $\mu=1,2,\dots,r$ ) in the case of  $U_r$  and, of  $C_S^S$  ( $S=1,2,3,4$ ) for  $U_4$ . From relations (38) it is easy to see that the sum of operators

$$\sum_{\mu=1}^r C_\mu^\mu = \text{Trace (of } U_p \text{ weight generators)}$$

commutes with all the generators of  $U_p$  and its subgroups

$U_{p-1}, U_{p-2}, \dots, U_1$ . Thus the eigenvalue of this sum is the

same for arbitrary state (90) as for state (94) of maximum weight in  $\mathcal{U}_p$ . For the latter, the eigenvalues of  $e_\mu^\mu$  ( $\mu=1,2,\dots,p$ ) are simply  $h_{\mu p}$  so that

$$\sum_{\mu=1}^p e_\mu^\mu \quad \text{has eigenvalue} \quad (h_{1p} + h_{2p} + \dots + h_{pp})$$

for any dimension  $p$ . Hence  $e_\mu^\mu$  has eigenvalue

$$w_\mu = (h_{1\mu} + h_{2\mu} + \dots + h_{\mu\mu}) - (h_{1,\mu-1} + h_{2,\mu-1} + \dots + h_{\mu-1,\mu-1}). \quad (94a)$$

Nagel & Vashinsky<sup>45)</sup> have recently constructed operators  $\Pi_r^k$  polynomial in the lowering generators  $e_\mu^{\mu'}$  ( $\mu' \leq \mu = 1, 2, \dots, r$ ) of a unitary group  $\mathcal{U}_r$  for any  $r$ . These lowering operators  $\Pi_r^k$  decrease by one unit the  $k^{\text{th}}$  representation label  $h_{kr-1}$  of the subgroup  $\mathcal{U}_{r-1}$ , keeping the representation of  $\mathcal{U}_{r-1}$  in maximum weight. Now, an  $N$ -particle totally antisymmetric  $\mathcal{U}_r$  representation  $[1^N 0^{4r-N}]$  will contain several, say  $x$ ,  $\mathcal{U}_r$  representations of highest weight. (See examples in Appendix A). Neglect for the moment the spin-isospin part of (94) ----- which is irrelevant for spin-isospin independent interactions and which is equivalent to assuming maximum spin and isospin projections  $M_s=S$ ,  $M_T=T$ , for any  $S$  and  $T$ , in the calculation of matrix elements of a function only of coordinates. Successive application of  $\Pi_r^k$  (for  $k=1, 2, \dots, r-1$ ) on each of the  $x$  immediately constructible states, like (79),

$$\left| \begin{array}{ccccccc} h_{1r} & h_{2r} & \dots & h_{kr} & \dots & h_{rr} \\ h_{1r} & h_{2r} & \dots & h_{kr} & \dots & h_{r-1r} \\ & & \dots & & & \\ & & & h_{1r} & h_{2r} & \\ & & & & h_{1r} & \end{array} \right. , \quad (95)$$

will give the  $x$  sets of kets with all the representations of  $\mathcal{U}_{r-1}$  in maximum weight contained in the  $x$  original  $\mathcal{U}_r$  representations of the type  $[h_{1r} h_{2r} \dots h_{rr}]$  and which satisfy the inequalities of the first two lines in (92). Next, by similar applications of  $\prod_{r-1}^k$  (for  $k=1,2,\dots,r-2$ ) one could generate all the states associated with all  $\mathcal{U}_{r-2}$  representation contained under each  $\mathcal{U}_{r-1}$  representation of maximum weight generated in the previous step, and which satisfied the inequalities of the 2nd and 3rd rows of (92). Continuing this process until one obtained all the representations contain in the  $\mathcal{U}_2$  representations, according now to the inequalities of the last two rows of (92), the full set of linearly independent polynomial functions transforming irreducibly with respect to the canonical  $\mathcal{U}_r$  chain of subgroups (89a) would be derived.

Removing now the restriction of spin-isospin independence, the complete basis with regard to the chain (89b) can also be constructed, though with more physical operators,<sup>46)</sup> as will be seen later.

One would therefore have all the  $N$ -particle totally anti-

symmetric states compatible with  $[1^N 0^{4r-N}]$  but in a "canonical" (or "mathematical") chain beginning with  $\mathcal{U}_r$ , i.e.,

$$|h_{1r}h_{2r}\dots h_{rr}, h_{1r-1}h_{2r-1}\dots h_{r-1r-1}, \dots, h_{12}h_{22}, h_{11}; \{V_4 V_{24} V_{34} V_{44}\} \beta SM_S TM_T \rangle \quad (96)$$

where  $\beta$  stands for a pair of hermitean operator eigenvalues needed to fully distinguish multiple occurrences of (ST) contained in the  $\mathcal{U}_4$  representation  $\{V_4 V_{24} V_{34} V_{44}\} = [h_{1r}h_{2r}\dots h_{rr}]$ . However, this basis (still non-physical in the orbital part) could be transformed to a scheme with definite total orbital angular momentum  $L$  and projection  $M_L$  with the aid of the matrix  $\|L^2\|$  calculated in the "canonical chain" basis (89a). The resulting complete set of totally antisymmetric N-particle states after this transformation would be:

$$| [f] \gamma L M_L; [\tilde{f}] \beta SM_S TM_T \rangle \quad (97)$$

$$\left\{ \begin{array}{l} [f] \equiv [h_{1r} h_{2r} \dots h_{rr}] \text{ (Young Partition)} \\ [\tilde{f}] \equiv [V_{14} V_{24} V_{34} V_{44}] = [h_{1r} h_{2r} \dots h_{rr}] \end{array} \right\} \quad (98)$$

provided the operator  $L_2$  were diagonal in the original basis (96), and this can be chosen to be so. Here,  $\gamma$  designates the set of quantum numbers required to distinguish multiple L-values contained in a given  $[f]$  representation. The (physical) chain of groups under which the complete set (97) transforms irreducibly is thus

$$\begin{array}{c}
 U_{4r} > U_r \times U_4 > \left\{ \begin{array}{l}
 U_r > R_3 > R_2(\text{orbital}) \\
 \uparrow \quad \uparrow \quad \uparrow \\
 [f] \quad (\gamma) \quad L \quad M_L \\
 \\
 U_4 > U_2 \times U_2 > R_2(\text{spin}) \times R_2(\text{isospin}) \\
 \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\
 \{f\} \quad (\beta) \quad S \quad T \quad M_S \quad M_T
 \end{array} \right. \quad (99)
 \end{array}$$

$\uparrow$   $[1^N 0^{4r-N}]$        $\uparrow$   $[f] \times \{f\}$

where the quantum number labels of the ket (97) are given underneath the chain; those with arrows pointing to a specific group are the labels for the irreducible representations of that group. The subchain  $R_3 > R_2$  represents the special case of spherical and axial symmetries discussed in Section 4. The group  $U_4$  of transformations in spin-isospin space contains the subgroup  $U_2 \times U_2$  of separate unitary transformations in 2-dimensional spinor and isospinor spaces. Each  $SU_2$  sub-group, as is well-known<sup>40)</sup>, is homomorphic to a group  $R_3$  (spin or isospin) of rotations in the (spin or isospin) space of three-dimensions spanned by the components

$$\begin{pmatrix} S_x, S_y, S_z \\ \text{or} \\ T_\xi, T_\eta, T_\xi \end{pmatrix}, \text{ which in turn contains the subgroup } R_2 \begin{pmatrix} \text{spin} \\ \text{or} \\ \text{isospin} \end{pmatrix}$$

of rotations around the  $\begin{pmatrix} z \\ \text{or} \\ \xi \end{pmatrix}$  axis. Example:

$$SU_2(\text{spin}) \approx R_3(\text{spin}) \supset R_2(\text{spin}) \quad (100)$$

$\uparrow$   
 $S$

$\uparrow$   
 $M_s$

The label  $\beta$  comes from hermitean operators formed from the generators of the  $U_4$  group which are to completely characterize the rows in  $U_4 \supset U_2 \times U_2$ . The label  $\gamma$  stands for irreducible representation labels associated with subgroups contained between

$U_r$  and  $R_\beta$  and also with hermitean operators which may be necessary to completely characterize the rows between two succeeding subgroups of the chain.

The problem of deducing the irreducible representations of  $R_\beta$  contained in a given one of  $U_r$  is a simple one. The same is true of the irreducible representations of  $U_2 \times U_2$  contained in a given  $U_4$  representation. In the former, one obtains the L-structure of a given N-particle Young diagram and in the latter, the S and T values contained in the conjugate N-particle young diagram are derived. For a given  $L$ , the different terms

$$2S+1, 2T+1 \quad L$$

arising from a representation  $\{V_1, V_2, V_3, V_4\}$  are called "supermultiplets". Jahn<sup>41)</sup> discusses the p and d orbits ( $U_3$  and  $U_5$ , respectively) and the (S,T)-structure for up to N=10 particles. For some illustrative examples, see Appendix A.

The generic state (97) can be L-S coupled to total angular

momentum  $J$  by the usual Clebsch-Gordan coefficient to give states

$$| [f] \gamma L ; [\tilde{f}] \rho S, T M_T ; J M_J \rangle = \sum_{M_L M_S} \langle L M_L M_S | J M_J \rangle | [f] \gamma L M_L ; [\tilde{f}] \rho S M_S, T M_T \rangle \quad (101)$$

Now, the total number of linearly independent anti-symmetric orbital-spin-isospin  $N$ -particle states is given by the simple result from statistical theory

$$\binom{4r}{N} = \frac{(4r)!}{(4r-N)! N!} \quad (102)$$

where  $4r$  is the number of single-particle states available. In the nuclear 2s-1d shell, for instance,  $r = \sum_k (2l_k + 1) = 1 + (2 \cdot 2 + 1) = 6$  orbital states. The number of orbital-spin-isospin single-particle states in this shell is  $6 \times 4 = 24$ . Then, for  $N = 1, 2, 3, 4, 5$  particles there are



N	# of totally anti-symmetric N-particle states
1	24
2	276
3	2,024
4	10,626
5	42,504

etc!

TABLE III.5.1

Obviously, one cannot hope to solve an N-body nuclear shell problem ---- even with a model hamiltonian ---- without making drastic reductions on the number of states to be considered as pertinent. One such criterion, first used by Wigner<sup>47)</sup>, is based on the fact that the attractivity of nuclear forces will favor the most symmetrical orbital configurations as lowest in energy. (This being the exact opposite of Hund's rule in atomic structure where forces are repulsive). For example, restricting 3-particle states in the 2s-1d shell

$$\begin{array}{rcl}
 \underline{U_{24}} & > & \underline{U_6 \times U_4} \quad \# \\
 [111] & > & [3] \times \{111\} \quad 224 \\
 & > & [21] \times \{21\} \quad 1400 \\
 & > & [111] \times \{3\} \quad \underline{400} \\
 & & \text{TOTAL: } 2,024 \text{ states}
 \end{array}
 \quad (103)$$

to the symmetric  $[3]$  partition of  $U_6$  would reduce the original number of states 2,024 to only 224. The  $U_4$  representation  $[111]$  is equivalent to  $[1]$  with  $(S,T) = (\frac{1}{2}, \frac{1}{2})$  (See Appendix A) so that  $2T+1 = 2$ , corresponding to  $M_T = \pm 1/2$ . For light nuclei coulomb effects are negligible so that one need only deal with half the 224 states, leaving 112. Moreover, our hamiltonian would certainly commute with  $\vec{J} = \vec{L} + \vec{S}$  so that instead of dealing with  $(2J+1)$  values of  $M_J$  for a given  $J$  one could limit the calculation to  $M_J = J$ . The 112 states are thus reduced to 17 states. This is still a large number resulting in still large matrices to be calculated and even further restrictions will be warranted. For this we turn to the simple harmonic oscillator well as a model for the nuclear common potential.

## 6.- Harmonic Oscillator Symmetry and the $U_3$ group.

### 6a. Single Particle.

High-energy electron scattering from nuclei<sup>48)</sup> indicate that while the nuclear common potential of heavy nuclei resemble a flat bottom shape with tapering edges of the Saxon-Woods well, that of light nuclei up to the Mg region resemble a harmonic oscillator parabolic well shape. The harmonic oscillator is simple to deal with analytically because of its group-symmetry properties. Let us therefore assume a hamiltontian

$$H = \sum_{i=1}^N H_i^{osc} + \sum_{i=1}^N H_i^{misc} + \sum_{i < j}^N V(r_{ij}) \quad (104)$$

$$H_i^{osc} = \frac{p_i^2}{2m} + \frac{1}{2} m \omega^2 r_i^2 \quad (105)$$

including any miscellaneous single-body interactions  $H_i^{misc}$  that may arise, viz. spin-orbit coupling, as well as residual inter-nucleon interactions  $\sum_{i < j} V(r_{ij})$ . The allowed energies and angular momenta a single particle are known to be given by

$$H_i^{osc} \psi_{n\ell m}(\vec{r}) = (v + \frac{3}{2}) \hbar \omega \psi_{n\ell m}(\vec{r})$$

$$L_i^2 \psi_{nlm}(\vec{r}) = l(l+1) \psi_{nlm}(\vec{r}) \quad (106 \text{ a, b, c})$$

$$L_{iz} \psi_{nlm}(\vec{r}) = m \psi_{nlm}(\vec{r})$$

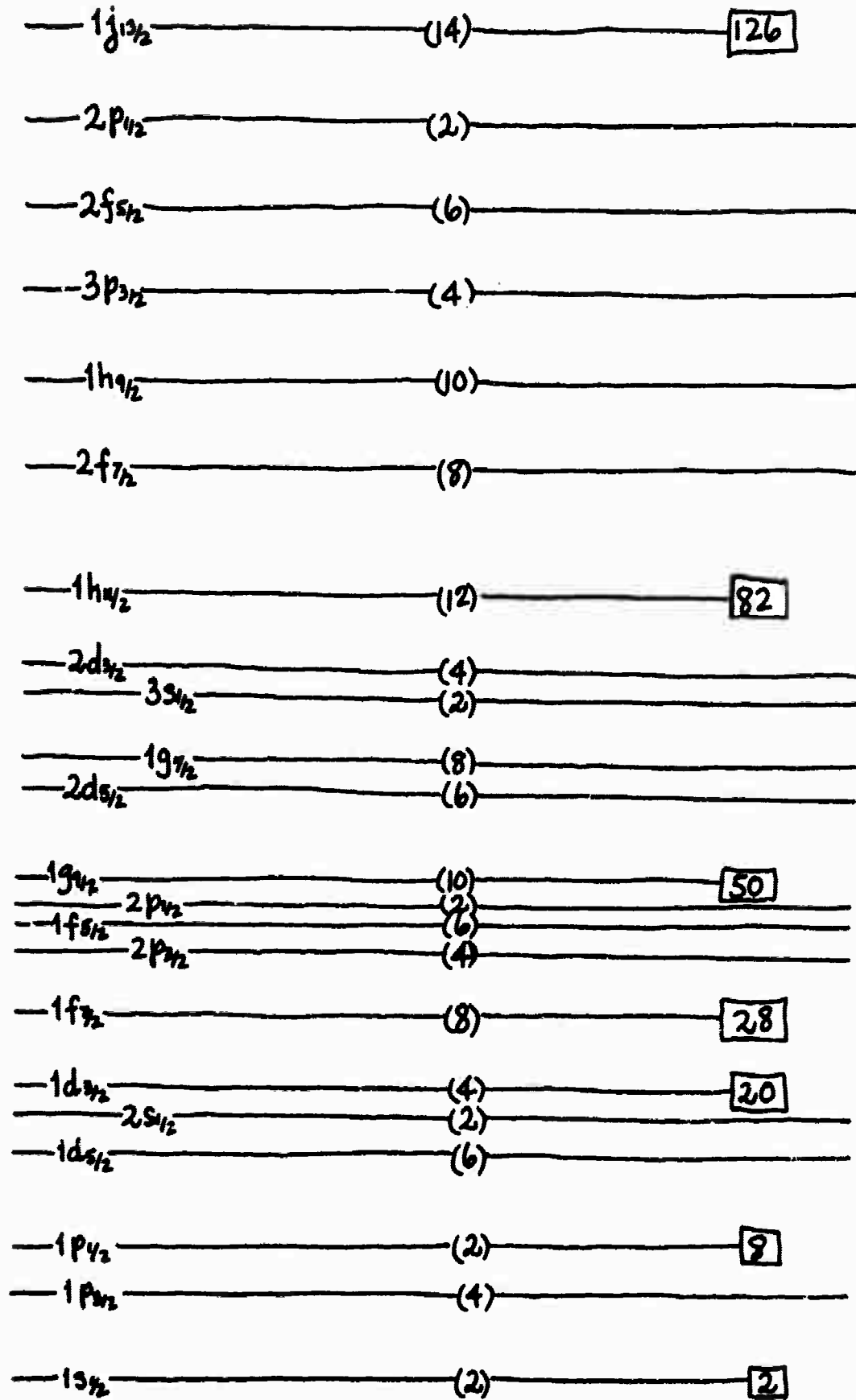
$$\psi_{nlm}(\vec{r}) = \frac{\sqrt{2(n!)}}{\sqrt{\Gamma(n+l+\frac{3}{2})}} e^{-r^2/2} L_n^{l+\frac{1}{2}}(r^2) r^l Y_{lm}(\theta, \varphi)$$

and with  $\nu \equiv 2n+l$  being the principal quantum number,  $n$  the radial quantum number in the Laguerre polynomial  $L_n^{l+\frac{1}{2}}(r^2)$ ,  $l$  and  $m$  the angular momentum and its projection. Since  $n$  is a non-negative integer,

$$l = \nu, \nu-2, \nu-4, \dots, 0 \quad (107)$$

and in general several orbitals (given  $l$ ) appear degenerate in a given shell (specified by  $\nu$ ). The energy difference between shells is  $\hbar\omega$ . The number of single-particle orbital states available within a given shell is

$$r = \sum_{l=0}^{\nu} (2l+1) = \frac{1}{2}(\nu+1)(\nu+2) \quad (108)$$



MAGIC NOS.

FIGURE III.6.1  
Doublet-splittings for neutrons.

so that  $2V$  neutrons or protons fill a given shell. Thus for  $V = 0$ , one has 2. For  $V = 1$ , there are 6; for  $V = 2$ , there are 12; for  $V = 3$ , 20, for  $V = 4$ , 30, etc. Hence, this model based solely on the harmonic oscillator well predicts the following neutron (or proton) numbers at each closure:

$V$	0	1	2	3	4	5	...
#	2	8	20	40	70	112	...

(109)

But, the empirical "magic numbers" at which nuclei show special stability in many regards are rather 2, 8, (14), 20, (28), 50, 82 and 126 ---- numbers in parenthesis referring to less pronounced stabilities. Thus only the first three fully-magic numbers 2, 8 and 20 are predicted in the scheme (109). By introducing a strong, attractive spin-orbit term to the oscillator hamiltonian Goeppert-Mayer<sup>49)</sup> and independently Haxel, Jensen & Suess<sup>50)</sup> predicted the magic and semi-magic numbers correctly, thanks to the ensuing doublet splitting (See Figure III.6.1).

The single-particle oscillator hamiltonian (105) and solutions (106) can, as is well-known, be formulated in terms of creation and annihilation boson operators.

Taking  $m = \hbar = \omega = 1$ , then

$$\begin{aligned}
 H_i^{osc} &= \frac{1}{2} (p^2 + r^2)_i \\
 &= \frac{1}{2} \sum_{q=1, \bar{1}, 0} (p_q p_q + x_q x_q)_i
 \end{aligned}
 \tag{110}$$

where  $x_{\pm 1} = \mp \frac{1}{\sqrt{2}}(x \pm iy)$ ,  $x_0 = z$ ;  $p_{\pm 1} = \mp \frac{1}{\sqrt{2}}(p_x \pm ip_y)$ ,  $p_0 = p_z$  (111)

are the spherical components of vectors  $\vec{r}$  and  $\vec{p}$ , the scalar products  $\vec{p} \cdot \vec{p}$  and  $\vec{r} \cdot \vec{r}$  in (110) being given with the usual metric, namely

$$\begin{aligned}\vec{r} \cdot \vec{r} &= r^2 = \sum_q (-)^q x_q x_{-q} = \sum_q x_q x_q \\ \vec{p} \cdot \vec{p} &= p^2 = \sum_q (-)^q p_q p_{-q} = \sum_q p_q p_q\end{aligned}\quad (112)$$

Furthermore, one defines the creation  $a_q^+$  and annihilation operators as the linear combinations

$$\begin{aligned}a_q^+ &\equiv \frac{1}{\sqrt{2}}(x_q - i p_q) \quad \text{and} \quad a_q \equiv \frac{1}{\sqrt{2}}(x_q + i p_q) \\ &\quad \text{or} \quad a_q = \frac{1}{\sqrt{2}}(x_q + i p_q)\end{aligned}\quad (113)$$

and since

$$\begin{aligned}[p_q, x_{q'}] &= [p_q, x_{q'}] = \frac{1}{i} \delta_{q, q'} \quad (q=1) \\ [p_q, p_{q'}] &= [x_q, x_{q'}] = 0\end{aligned}\quad (114)$$

one obtains the relations

$$\begin{aligned}[a_q, a_{q'}^+] &= \delta_{q, q'} \\ [a_q, a_{q'}] &= [a_q^+, a_{q'}^+] = 0\end{aligned}\quad (115a, b)$$

indeed the commutation relations for boson operators. Raising and lowering of the indices is governed by the metric in (112) so that

$$a^{+q} = (-)^q a_{-q}^+, \quad a_q = (-)^q a^{-q}, \quad (a^q)^+ = a_q^+. \quad (116)$$

Suppressing the particle index  $i$  in (110), it follows from (112) and (113) that

$$H^{osc} = \sum_q a_q^+ a^q + \frac{3}{2} \equiv \sum_q c_q^q + \frac{3}{2}. \quad (117)$$

Similary, it can be seen that

$$\begin{aligned} L_q &= (\vec{r} \times \vec{p})_q = \sum_{q'q''} \epsilon_{qq'q''} a^{+q'} a^{q''} \\ &= \sum_{q'q''} (-)^{q'} \epsilon_{q-q'q''} a_q^+ a^{q''} \\ L_q &= \sum_{q'q''} (-)^{q'} \epsilon_{q-q'q''} c_{q'}^{q''} \end{aligned} \quad (118)$$

the  $\epsilon_{qq'q''}$  being the completely antisymmetric tensor,  $\epsilon_{qq'q''} = 1$  if  $q q' q''$  are cyclic in order 120,  $\epsilon_{qq'q''} = -1$  if not cyclic and  $\epsilon_{qq'q''} = 0$  if any two indices repeat. In the step previous to the last in (118) the index  $q'$  was lowered according to rules (116).

As early as 1940, it was noted<sup>51)</sup> that a harmonic oscillator potential remained invariant under the transformations of the three-dimensional unitary group  $U_3$ . In (117) and (118) we defined the single-particle operators



$$C_q^{q'} \equiv a_q^+ a_q^{q'} \quad (q, q' = 1, \bar{1}, 0) \quad (119)$$

which, upon using relations (115a, b), can be seen to obey the commutation relations

$$[C_q^{q'}, C_{q''}^{q'''}] = C_q^{q'''} \delta_{q''}^{q'} - C_{q''}^{q'} \delta_q^{q'''} \quad (120)$$

which ---- by the same arguments associated with (32) ---- stand for the Lie commutator algebra of the  $3^2=9$  generators for infinitesimal transformations of the group  $U_3$ . That this is the symmetry group of  $H^{osc}$  of (117) follows from the fact that  $H$  commutes with all the generators of  $U_3$ , that is

$$\begin{aligned} [H^{osc}, C_{q'}^{q''}] &= \sum_q [C_q^q, C_{q'}^{q''}] \\ &= \sum_q \{ C_q^{q''} \delta_{q'}^q - C_{q'}^q \delta_q^{q''} \} \\ &= 0. \end{aligned}$$

Again, as in (72) the set  $\{C_q^{q'}\}$  of  $U_3$  generators can be subdivided into three classes: lowering  $C_1^1, C_0^{\bar{1}}, C_0^0$ ; weight-giving  $C_1^1, C_0^{\bar{1}}, C_0^0$  (in that order), and raising  $C_1^{\bar{1}}, C_1^0$  and  $C_1^0$ . Moreover,  $R_3$  being a subgroup of  $U_3$ , the generators of the former ( $L_q$  with  $q = 1, \bar{1}, 0$ ) should be expressible as linear combinations of those of  $U_3$  and from (118) this is indeed the case. Operators  $L^2$  and  $L_z = L_0$  can be simultaneously diagonalized along with  $H^{osc}$  since

$$[H^{\text{osc}}, L^2] = [H^{\text{osc}}, \sum_q (-)^q L_q L_{-q}] = 0$$

$$[H^{\text{osc}}, L_0] = 0 \quad (122)$$

$$[L^2, L_0] = 0.$$

Now, an arbitrary single-particle harmonic oscillator state can be written as

$$|n, n_{\bar{1}}, n_0\rangle = \frac{(a_1^+)^{n_1} (a_{\bar{1}}^+)^{n_{\bar{1}}} (a_0^+)^{n_0}}{\sqrt{n_1! n_{\bar{1}}! n_0!}} |\bar{0}\rangle \quad (123)$$

in terms of the numbers  $n_q$  of oscillator quanta along the three spherical "directions"  $q = 1, \bar{1}, 0$ . The state  $|\bar{0}\rangle$  is the oscillator ground state (no excitation) and the radical provides normalization. There being  $r = \frac{1}{2}(\nu+1)(\nu+2)$  different possibilities for the triplet  $(n, n_{\bar{1}}, n_0)$  where  $n_1 + n_{\bar{1}} + n_0 = \nu$  always, the vector (123) spans the  $r$ -dimensional orbital subspace of our problem. From (115a) one notices that the effect of  $a_q^+$  acting on any homogeneous polynomial  $P(a_q^+)$  is equivalent to a partial derivative:

$$a_i^\dagger P(a_i^\dagger) = \frac{\partial}{\partial a_i^\dagger} P(a_i^\dagger). \quad (124)$$

The effect of  $H^{osc}$  and  $L_z$  upon  $|n_i, n_{\bar{i}}, n_o\rangle$  is thus immediate:

$$\begin{aligned} H^{osc} |n_i, n_{\bar{i}}, n_o\rangle &= \left\{ \sum_i a_i^\dagger \frac{\partial}{\partial a_i^\dagger} + \frac{3}{2} \right\} \frac{(a_i^\dagger)^{n_i} (a_{\bar{i}}^\dagger)^{n_{\bar{i}}} (a_o^\dagger)^{n_o}}{\sqrt{n_i! n_{\bar{i}}! n_o!}} |0\rangle \\ &= (n_i + n_{\bar{i}} + n_o + \frac{3}{2}) |n_i, n_{\bar{i}}, n_o\rangle \\ &= (v + \frac{3}{2}) |n_i, n_{\bar{i}}, n_o\rangle \end{aligned} \quad (125)$$

$$\begin{aligned} L_z |n_i, n_{\bar{i}}, n_o\rangle &= (c_i^\dagger - c_{\bar{i}}^\dagger) |n_i, n_{\bar{i}}, n_o\rangle \\ &= (n_i - n_{\bar{i}}) |n_i, n_{\bar{i}}, n_o\rangle \\ &\equiv m |n_i, n_{\bar{i}}, n_o\rangle \end{aligned} \quad (126)$$

These results are identical with (106a,c) in the sense that the energy and the angular momentum projection are diagonal. In the present case,  $L^2$  is not diagonal as it was in the old sets

$\psi_{k,q}(\vec{r})$  which forms a basis for an irreducible representation according to the chain of single-particle transformation groups

$$U_3 \supset R_3 \supset R_2 \quad (127)$$

with representation labels  $v$ ,  $\ell$  and  $m$ , respectively. Dealing with this physical chain is more difficult at present than working

with the canonical (or mathematical) chain

$$U_3 \supset U_2 \supset U_1 \quad (128)$$

for which the basis  $|n, n_T, n_0\rangle$  is most appropriate. Specifically, the group, its weight-giving operators and the corresponding representation labels are simply

$$U_3: c_i, c_i^\dagger, c_0 \longrightarrow (n, n_T, n_0)$$

$$U_2: c_i, c_i^\dagger \longrightarrow (n, n_T)$$

$$U_1: c_i \longrightarrow (n_i)$$

and, in the Gel'fand notation, a general state forming the basis transforming irreducibly according to (128) is

$$|n, n_T, n_0\rangle = \left| \begin{array}{ccc} n, & n_T & n_0 \\ & n, & n_T \\ & & n_i \end{array} \right\rangle. \quad (129)$$

#### 6b. Many Particles.

The operators  $c_i^{q'}$  defined in (119) are single-particle operators. We are now interested in dealing with the N-particle operator

$$\sum_{i=1}^N c_i^{q'}(i).$$

Being spin and isospin independent we can use formula (35) to cast (130) into second-quantization formalism:

$$C_q^{q'} = \sum_{\mu\mu'} \langle \mu' | C_q^{q'} | \mu \rangle b_{\mu'}^{\mu} \quad (131)$$

recalling from (37) that

$$b_{\mu'}^{\mu} = \sum_s b_{\mu's}^+ b^{s\mu} \quad \left( \begin{matrix} \mu, \mu' = 1, 2, \dots, r \\ s \in \sigma = 1, 2, 3, 4 \end{matrix} \right) \quad (132)$$

are the group generators of  $\mathcal{U}_r$ , the group of transformations in orbital space, and where  $\mu$  stood for the single-particle quantum numbers  $\nu l m$ , to be replaced here by the new set  $n, n_T, n_o$ .

Thus,

$$C_q^{q'} = \sum_{n, n_T, n_o} \sum_{n', n'_T, n'_o} \langle n', n'_T, n'_o | C_q^{q'} | n, n_T, n_o \rangle b_{n', n'_T, n'_o}^{n, n_T, n_o} \quad (133)$$

The coefficients in this expansion are very easily found from definitions (119) and (124) whose effect on the normalized oscillator state (123) is seen to give the simple result (see Appendix G for details):

$$C_q^{q'} = \sum_{n, n_T, n_o} \sqrt{n_q (n_q - \delta_{q,q'} + 1)} b_{n+\delta_{q,1}-\delta_{q,1}, n_T+\delta_{q,T}-\delta_{q,T}, n_o+\delta_{q,0}-\delta_{q,0}}^{n, n_T, n_o} \quad (134)$$

These 9 operators, from (131) and relations (120), obey the commutation relations

$$[C_q^{q'}, C_{q''}^{q''}] = \sum_{\mu\mu'} \langle \mu' | [C_q^{q'}, C_{q''}^{q''}] | \mu \rangle b_{\mu'}^{\mu}$$

$$= \left( \sum_{\mu\mu'} \langle \mu' | C_q^{q''} | \mu \rangle b_{\mu'}^{q''} \right) \delta_q^{q'} - \left( \sum_{\mu\mu'} \langle \mu' | C_q^{q'} | \mu \rangle b_{\mu'}^{q'} \right) \delta_q^{q''}$$

$$[C_q^{q'}, C_q^{q''}] = C_q^{q''} \delta_q^{q'} - C_q^{q'} \delta_q^{q''} \quad (135)$$

which compose the Lie algebra of  $U_3$ . For the particular case of the 2s-1d shell

$$\eta_1 + \eta_{\bar{1}} + \eta_0 = \nu = 2 \text{ quanta}; \quad \nu = \frac{1}{2}(\nu+1)(\nu+2) = 6 \quad (136)$$

$$U_r = U_6 \supset U_3.$$

Expansion (134) of the generators of the subgroup  $U_3$  in terms of those of the group  $U_6$  can be explicitly given if we first take the convention of enumerating the 6 states  $\mu = (\eta, \eta_{\bar{1}}, \eta_0)$  in order of decreasing weights (in the sense of 72) and calling  $q = 1, \bar{1}, 0 \rightarrow 1, 2, 3$  for convenience:

$\mu$	$\eta, \eta_{\bar{1}}, \eta_0$
1	2 0 0
2	1 1 0
3	1 0 1
4	0 2 0
5	0 1 1
6	0 0 2

and

$$\begin{aligned} C_1^1 &\rightarrow C_1^1 \\ C_{\bar{1}}^{\bar{1}} &\rightarrow C_2^2 \\ C_0^0 &\rightarrow C_3^3 \\ C_1^0 &\rightarrow C_1^3 \\ &\text{etc} \end{aligned} \quad (137)$$

Therefore, we have the set of  $U_3$  generators  $\{C_q^{q'}\}$  expressed as:

$$\text{RAISING} \begin{cases} C_1^2 = \sqrt{2} b_1^2 + \sqrt{2} b_2^4 + b_3^5 \\ C_1^3 = \sqrt{2} b_1^3 + \sqrt{2} b_3^6 + b_2^5 \\ C_2^3 = b_2^3 + \sqrt{2} b_4^5 + \sqrt{2} b_5^6 \end{cases}$$

$$\text{WEIGHT} \begin{cases} C_1^1 = 2b_1^1 + b_2^2 + b_3^3 \\ C_2^2 = b_2^2 + 2b_4^4 + b_5^5 \\ C_3^3 = b_3^3 + b_5^5 + 2b_6^6 \end{cases} \quad (138)$$

$$\text{LOWERING} \begin{cases} C_2^1 = \sqrt{2} b_2^1 + \sqrt{2} b_4^2 + b_5^3 \\ C_3^1 = \sqrt{2} b_3^1 + \sqrt{2} b_6^3 + b_5^2 \\ C_3^2 = b_3^2 + \sqrt{2} b_5^4 + \sqrt{2} b_6^5 \end{cases}$$

The N-particle interactionless oscillator hamiltonian

$$\sum_{i=1}^N H_i^{\text{osc}} = \sum_{i=1}^N \left\{ \sum_q C_q^q + \frac{3}{2} \right\} \quad (136)$$

can likewise be obtained, from (35), in second-quantization language as

$$\begin{aligned}
 \mathcal{H}^{osc} &= \sum_{\mu\mu'} \langle \mu | \sum_{\mathbf{q}} C_{\mathbf{q}}^{\mathbf{q}} + \frac{3}{2} | \mu' \rangle C_{\mu}^{\mu'} \\
 &= \sum_{\mathbf{q}} C_{\mathbf{q}}^{\mathbf{q}} + \frac{3}{2} \sum_{\mu} C_{\mu}^{\mu} \\
 &= (\eta_+ + \eta_- + \eta_0 + 3/2) \sum_{\mu} C_{\mu}^{\mu} \\
 &= (v + 3/2) \mathcal{N}
 \end{aligned} \tag{139}$$

$\mathcal{N}$  being the number operator defined in (71). Similarly, the N-particle momentum operator in (118)

$$\sum_{i=1}^N L_{\mathbf{q}}(i) = \sum_{i=1}^N \sum_{\mathbf{q}'\mathbf{q}''} (-)^{\mathbf{q}'} \epsilon_{\mathbf{q}-\mathbf{q}'\mathbf{q}''} C_{\mathbf{q}'}^{\mathbf{q}''}(i) \tag{140}$$

now becomes

$$\begin{aligned}
 L_{\mathbf{q}} &= \sum_{\mathbf{q}'\mathbf{q}''} (-)^{\mathbf{q}'} \epsilon_{\mathbf{q}-\mathbf{q}'\mathbf{q}''} \left\{ \sum_{\mu\mu'} \langle \mu | C_{\mathbf{q}'}^{\mathbf{q}''} | \mu' \rangle C_{\mu}^{\mu'} \right\} \\
 L_{\mathbf{q}} &= \sum_{\mathbf{q}'\mathbf{q}''} (-)^{\mathbf{q}'} \epsilon_{\mathbf{q}-\mathbf{q}'\mathbf{q}''} C_{\mathbf{q}'}^{\mathbf{q}''} \quad (\mathbf{q} = +, -, 0)
 \end{aligned} \tag{141}$$

so that explicitly, the spherical components

$$L_+ = -(C_0^+ + C_1^0), \quad L_0 = C_1^+ - C_1^-, \quad L_- = C_0^+ + C_1^- \tag{142a}$$

are easily seen to obey the commutation relations

$$[L_0, L_{\pm 1}] = \pm L_{\pm 1} \quad [L_+, L_-] = -L_0 \tag{142b}$$



of the group  $R_3 \subset U_3$ .

In the many-particle case, the group  $U_3$  with generators  $\{C_q^{q'}\}$  ( $q, q' = 1, \bar{1}, 0 \rightarrow 1, 2, 3$ ) obviously continues to be the symmetry group of the harmonic oscillator as its hamiltonian (136) is a  $U_3$  group in variant, namely,

$$[H^{osc}, C_q^{q'}] = 0. \quad (143a)$$

Hence, just as  $L^2$  (in section 4) commuting with the  $R_3$  generators  $L_q$  ( $q = +, 0, -$ ) lead to its diagonality (55a) in a base  $P_{lm}$  transforming irreducibly under  $R_3$ , we here have that  $H^{osc}$  will be diagonal in a base irreducible under  $U_3$  (See Appendix ). As in (73 a,b) for the group  $U_r$ , the maximum weight polynomial belonging to the  $U_3$  basis set is defined by

$$\begin{aligned} C_q^q P_{(h_1, h_2, h_3)} &= h_{q3} P_{(h_1, h_2, h_3)} \\ C_q^{q'} P_{(h_1, h_2, h_3)} &= 0 \end{aligned} \quad \begin{matrix} (q \rightarrow 1, 2, 3) \\ q < q' \end{matrix} \quad (144a, b)$$

where  $C_q^q$  and  $C_q^{q'} (q < q')$  are respectively the weight-giving and raising operators and the second index on the h's refers to  $U_3$ . The raising operators (144b) of  $U_3$  are linear in  $U_r$  raising operators according to (138). If  $P$  is chosen with maximum weight in  $U_r$  as in prescription (79), it will thus also be of maximum weight in  $U_3$ . The full irreducible basis in  $U_3$  could be obtained using lowering operators of  $U_3$  (the  $L_r^k$ ) as was mentioned in the previous section for  $U_r$ . This would give the basis in the  $U_3 \supset U_2 \supset U_1$  chain, later to be transformed to one in the physical chain  $U_3 \supset R_3 \supset R_2$ .

The second-quantization version of the N-particle hamiltonian (104) will thus be

$$\mathcal{H} = \mathcal{H}^{osc} + \mathcal{H}_{int} \quad (143b)$$

with  $\mathcal{H}_{int}$  including both single- and two-body interactions to be dealt with later.

Conclusion: in the  $\mathcal{U}_r \supset R_3 \supset R_2$  section of the chain of groups (99) with respect to which our many-particle nuclear states are to transform irreducibly, one can insert the extra (oscillator) symmetry group so that

$$\mathcal{U}_r \supset U_3 \supset R_3 \supset R_2 \quad (145a)$$

$$\begin{array}{ccccccc} & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ | [f] \propto (h_{13} h_{23} h_{33}) \omega L M_L \rangle & & & & & & & \end{array} \quad (145b)$$

will provide the additional classification quantum numbers  $(h_{13} h_{23} h_{33})$  of  $U_3$  needed to further distinguish multiple  $L$ -values appearing under a given  $[f]$  partition of  $\mathcal{U}_r$ . Labels  $\alpha$  and  $\omega$ , not proceeding from any particular group, serve respectively to distinguish multiple  $(h_{13} h_{23} h_{33})$  values under a given  $[f]$  and multiple  $L$ -values under a given  $(h_{13} h_{23} h_{33})$ . The basis state irreducible under (145) is thus designated as:

$$\begin{array}{ccccccc} | [f] \propto (h_{13} h_{23} h_{33}) \omega L M_L \rangle & & & & & & (145c) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \text{PERMUTATIONAL} & \text{HARMONIC OSCILLATOR} & \text{SPHERICAL} & \text{AXIAL} \\ \text{SYMMETRY} & \text{SYMMETRY} & \text{SYMMETRY} & \text{SYMMETRY} \end{array}$$

CHAPTER

IV. MODEL RESIDUAL INTERACTIONS AND THEIR GROUP SYMMETRIES.

In the shell model one is dealing with a hamiltonian of the type

$$H = \sum_i \frac{p_i^2}{2m} + \sum_i U(r_i) + \xi \sum_i \vec{l}_i \cdot \vec{\lambda}_i \quad (147)$$

$$+ \sum_{i,j} V(r_{ij}) (W + M P_{ij}^r - H P_{ij}^\tau + B P_{ij}^\sigma)$$

where  $1$ ,  $P_{ij}^r$ ,  $P_{ij}^\tau$  and  $P_{ij}^\sigma$  are the Wigner, Majorana, Heisenberg and Bartlett exchange operators<sup>52)</sup> defined by

$$P_{ij}^\sigma \equiv \frac{1}{2} (1 + \vec{\sigma}_i \cdot \vec{\sigma}_j) \quad P_{ij}^\tau \equiv \frac{1}{2} (1 + \vec{\tau}_i \cdot \vec{\tau}_j) \quad (148a, b, c)$$

$$P_{ij}^r \equiv -\frac{1}{4} (1 + \vec{\sigma}_i \cdot \vec{\sigma}_j) (1 + \vec{\tau}_i \cdot \vec{\tau}_j)$$

and  $W, M, H$  and  $B$  indicate the magnitudes of those components in the exchange mixture.

We shall assume that  $U(r_i)$  is a harmonic oscillator common potential

$$U(r_i) = \frac{1}{2} m \omega^2 r_i^2$$

and furthermore restrict ourselves to particles interacting within a single shell of this harmonic oscillator. Under this assumption

the two-body interaction can be analyzed as a mixture of long- and short-range correlations with definite group theoretical properties.

1. Long Range Central Interaction: The  $Q^2$  force.

Let us take an arbitrary (say, Gaussian) form of the central potential function in (147):

$$V(r_{ij}) = -V_0 e^{-r_{ij}^2/a^2} \quad (i, j = 1, 2, \dots, N) \quad (151)$$

having variable well-depth  $V_0$  and range  $a$  parameters. In addition to short-ranged forces ( $\sim 1$  to 2 fermis) within the nucleus there are longer-ranged correlations ( $\sim$  nuclear radius) responsible for collective behaviour of the rotational and vibrational types.<sup>6, 7)</sup> Assuming a gaussian well, rather flat throughout most of the nuclear volume (large  $a$ ), we can expand the exponential in (151) retaining only the first few terms, namely

$$V(r_{ij}) = -V_0 \left[ 1 - \frac{r_{ij}^2}{a^2} + \frac{r_{ij}^4}{2a^4} - \dots \right]$$

where the relative distance  $r_{ij} = \sqrt{r_i^2 + r_j^2 - 2\vec{r}_i \cdot \vec{r}_j}$  can be inserted above, and upon convenient rearrangements obtain

$$\begin{aligned} V(r_{ij}) = -V_0 \left\{ 1 - \frac{r_i^2 + r_j^2}{a^2} + \frac{2\vec{r}_i \cdot \vec{r}_j}{a^2} + \frac{r_j^4 + r_i^4}{2a^4} \right. \\ \left. + \frac{5r_i^2 r_j^2}{3a^4} - \frac{2(r_i^2 + r_j^2)\vec{r}_i \cdot \vec{r}_j}{a^4} + \frac{2[(\vec{r}_i \cdot \vec{r}_j)^2 - \frac{1}{3}r_i^2 r_j^2]}{a^4} + \dots \right\} \end{aligned} \quad (152)$$

the energy spectrum is determined by matrix elements of the type  $\langle n_i l_i, n_f l_f, LM | V(r_{if}) | n_i' l_i', n_f' l_f', LM \rangle$  for excitations within a single oscillator shell  $\nu$ . But, from (105) and (106) one deduces that the 1st, 2nd, and 5th terms of (152) contribute only as functions of  $\nu = 2n + l$ . Terms linear in  $(\vec{r}_i \cdot \vec{r}_f)$  ----- 3rd and 6th ----- give zero because states  $| n_i' l_i' \rangle$  and  $\langle n_i l_i |$  have the same parity while  $\vec{r}_i, \vec{r}_f$  have odd parity. The 4th term is separated in the particle coordinates and thus contributes not as an interaction but as a correction to the common potential. Therefore, the only pertinent term remaining for large  $a$  is the 7th:

$$V(r_{if}) \approx - \frac{2V_0}{a^4} [(\vec{r}_i \cdot \vec{r}_f)^2 - \frac{1}{3} r_i^2 r_f^2] \quad (153)$$

This term is related to the harmonic oscillator symmetry group  $U_3$  as we shall see.

The creation  $a_q^+$  and annihilation  $a_q$  boson operators defined in (113) for harmonic oscillator states (123) transform like contravariant vector components, i.e., like a wave function of orbital angular momentum  $l=1$ . Let us vector-couple them together to form the k-rank, q-projection tensors

$$\begin{aligned} U_q^{(k)} &= \sum_{q'q''} \langle 11 q' q'' | kq \rangle a_{q'}^+ a_{q''} \\ &= \sum_{q'q''} (-)^{q''} \langle 11 q' - q'' | kq \rangle C_{q'}^{q''} \end{aligned} \quad (154)$$

since  $C_{q'}^{q''} \equiv a_{q'}^+ a_{q''}$  with  $q', q'' = 1, \bar{1}, 0$  are the 9 single-particle generators of  $U_3$  defined in (119) and which obey the Lie algebra

(120). In linear combinations (154) only  $k=0,1,2$  and  $q=-k,-k+1,\dots,k$  can occur so that there are

$$\sum_{k=0,1,2} (2k+1) = 1+3+5 = 9$$

linearly independent operators  $U_q^{(k)}$  which can equivalently be considered the 9 generators of  $U_3$ . (It can easily be seen that the commutators between these 9 operators also form a closed algebra.) In particular,

$$\underline{k=2} \quad U_q^{(2)} = \sum_{q'q''} (-)^{q''} \langle 11q'-q'' | 2q \rangle C_{q'}^{q''} \quad (155)$$

$$\begin{aligned} \underline{k=1} \quad U_q^{(1)} &= \sum_{q'q''} (-)^{q''} \langle 11q'-q'' | 1q \rangle C_{q'}^{q''} \\ &= -\frac{1}{\sqrt{2}} \sum_{q'q''} (-)^{q'} \epsilon_{q-q'q''} C_{q'}^{q''} = -\frac{1}{\sqrt{2}} L_q \end{aligned} \quad (156)$$

$$\begin{aligned} \underline{k=0} \quad U_q^{(0)} &= \sum_{q'q''} (-)^{q'} \langle 11q'-q' | 00 \rangle C_{q'}^{q''} \\ &= -\frac{1}{\sqrt{3}} \sum_{q'} C_{q'}^{q'} = -\frac{1}{\sqrt{3}} H_0 \end{aligned} \quad (157)$$

where (156) follows from definitions (118), and (157) save a scaling constant is the single-particle oscillator hamiltonian of (117).

Considering the 5 operators for  $k=2$  we have from (113) giving the

$a_q^+ a_q$  in terms of coordinates and momenta,

$$U_q^{(2)} = \sum_{q'q''} \langle 11q'q'' | 2q \rangle a_{q'}^+ a_{q''}$$

$$\begin{aligned}
 &= \sum_{q'q''} \langle 11q'q'' | 2q \rangle [x_{q'}, x_{q''} + p_{q'}, p_{q''}] \\
 &= \frac{1}{2} \left\{ \sqrt{\frac{8\pi}{15}} r^2 Y_{2q}(\theta_r, \varphi_r) + \sqrt{\frac{8\pi}{15}} p^2 Y_{2q}(\theta_p, \varphi_p) \right\}
 \end{aligned}$$

given as solid spherical harmonics in coordinate and momentum spaces. The matrix element, between oscillator states of a given shell, of a momentum function is identical with that of the same function of coordinates ----- thus

$$U_q^{(2)} \sim \sqrt{\frac{8\pi}{15}} r^2 Y_{2q}(\theta, \varphi) \quad (158)$$

The mass quadrupole moment induced by a single particle is conventionally<sup>53)</sup> taken as the expectation value of the operator

$$Q_0 \equiv 3z^2 - r^2 = \sqrt{\frac{16\pi}{5}} r^2 Y_{20}(\theta, \varphi) \quad (159)$$

so that the 5  $U_3$  operators

$$Q_q = \sqrt{6} U_q^{(2)} \quad (160)$$

can be thought of as representing generators of infinitesimal quadrupole distortions. A quadrupole-quadrupole interaction between the  $i^{\text{th}}$  and  $j^{\text{th}}$  particles is simply the scalar product

$$Q_{ij}^2 \equiv \sum_q (-1)^q Q_q(i) Q_{-q}(j) \quad (161)$$

From (158) taken separately for particles  $i$  and  $j$ , and the addition theorem for spherical harmonics,<sup>54)</sup>

$$\begin{aligned} Q_{ij}^2 &= \frac{16\pi}{5} r_i^2 r_j^2 \sum_q (-)^q Y_{2q}(i) Y_{2-q}(j) \\ &= 4 r_i^2 r_j^2 P_2(\cos \omega_{ij}) \\ &= 4 r_i^2 r_j^2 P_2\left(\frac{\vec{r}_i \cdot \vec{r}_j}{r_i r_j}\right) \\ &= 2 r_i^2 r_j^2 \left[ 3 \frac{(\vec{r}_i \cdot \vec{r}_j)^2}{(r_i r_j)^2} - 1 \right] \end{aligned}$$

$$Q_{ij}^2 = 6 \left[ (\vec{r}_i \cdot \vec{r}_j)^2 - \frac{1}{3} r_i^2 r_j^2 \right] \quad (162)$$

which is identical in form to (153) ----- the only significant portion of a long-ranged central potential.

The expressibility of (162) in terms of  $U_3$  generators is the crux of this whole section. The Casimir operator of  $R_3$  defined in (56) as

$$L^2 = \sum_q (-)^q L_q L_{-q} = \sum_q L_q L^q \quad (163)$$

was particularly useful in solving the asymmetric top problem discussed in the Introduction: its eigenvalue was simply  $l(l+1)$ ,



dependent only on the irreducible representation label and regardless of its row. A contraction analogous to (163) yields

$$G_{ij} \equiv \sum_{q'q''} C_{q'}^{q''}(i) C_{q''}^{q'}(j) \quad (i, j = 1, 2, \dots, N) \quad (164a)$$

Then,

$$\sum_{i < j} G_{ij} \quad (164b)$$

is the (second degree) Casimir operator of  $U_3$  which should and does commute with all the many-particle  $U_3$  operators  $\sum_i C_{q'}^{q''}(i)$ . Employing the orthonormality relations between Clebsch-Gordan coefficients<sup>35)</sup> one can reverse expansions (154) so that

$$C_{q'}^{q''} = (-)^{q''} \sum_{kq} \langle 11q' - q'' | kq \rangle \mathcal{U}_q^{(k)} \quad (165)$$

and which is valid separately for  $i$  and  $j$ . Using this expression, the Casimir operator (164b) will contain

$$\begin{aligned} G_{ij} &= \sum_k \left\{ \sum_q (-)^q \mathcal{U}_q^{(k)}(i) \mathcal{U}_{-q}^{(k)}(j) \right\} \\ &= \sum_q (-)^q \left\{ \mathcal{U}_q^{(0)}(i) \mathcal{U}_{-q}^{(0)}(j) + \mathcal{U}_q^{(1)}(i) \mathcal{U}_{-q}^{(1)}(j) + \mathcal{U}_q^{(2)}(i) \mathcal{U}_{-q}^{(2)}(j) \right\} \\ &= \frac{1}{3} H_0^2(i) + \frac{1}{2} \sum_q (-)^q L_q(i) L_{-q}(j) + \sum_q (-)^q Q_q(i) Q_{-q}(j) \end{aligned}$$

having recalled (156), (157) and (160). Finally

$$G_{ij} = Q_{ij}^2 + \frac{1}{2} L_{ij}^2 + \frac{1}{3} H_{oi}^2 \quad (166a, b)$$

$$Q_{ij}^2 = G_{ij} - \frac{1}{2} L_{ij}^2 - \frac{1}{3} H_{oi}^2.$$

Clearly, since the interaction operator  $\sum_{i,j} Q_{ij}^2$  is given solely in terms of the Casimir operators of  $U_3$  and  $R_3$ , and in terms of  $\sum_i H_{oi}$ , it will be diagonal in a basis for irreducible representations of the group  $U_3 \supset R_3$ .

To cast the N-particle mass quadrupole moment operator  $\sum_{i=1}^N Q_q(i)$  into second-quantization formulism we again use formula (35) to give the linear combination

$$\begin{aligned} Q_q &= \sum_{\mu, \mu'} \langle \mu | Q_q | \mu' \rangle b_{\mu}^{\mu'} \\ &= \sum_{\mu, \mu'} \langle \mu | \sqrt{\frac{16\pi}{5}} r^2 Y_{2q}(\theta, \varphi) | \mu' \rangle b_{\mu}^{\mu'} \end{aligned} \quad (167)$$

whose expectation value, for  $q=0$ , between appropriate states is the quadrupole moment induced by N particles. (More detail concerning quadrupole moments and transition rates are given in Appendix J ).

Result (166b) in the new many-particle formulism will be

$$Q^2 = G - \frac{1}{2} L^2 - \frac{1}{3} H_0^2 \quad (168)$$

where the many-body Casimir operator  $\mathcal{G}$  is constructed as in (164) but now in terms of the many-body  $U_3$  generators defined in (131):

$$\mathcal{G} = \sum_{q, q'} C_q^{q'} C_{q'}^q, \quad [\mathcal{G}, C_q^{q'}] = 0. \quad (169)$$

Moreover, we shall have

$$\mathcal{Q}^2 = \sum_q (-)^q \mathcal{Q}_q \mathcal{Q}_{-q} \quad (170)$$

$$\mathcal{L}^2 = \sum_q (-)^q \mathcal{L}_q \mathcal{L}_{-q} \quad (171)$$

$$\mathcal{H}_0 = \mathcal{H}^{osc} - \frac{3}{2} \mathcal{N} = \nu \mathcal{N} \quad (172)$$

in accordance with results (139), (141) and (167).

Our model long-range interaction (168) is diagonal under  $U_3 \supset R_3$  as argued above. To find its spectrum is a simple matter precisely because of its group symmetry. The eigenvalues of  $\mathcal{L}^2$  and  $\mathcal{H}_0$  are simply  $L(L+1)$  and  $\nu N$ , respectively,  $L$  being the total orbital angular momentum of a given state. Now,  $\mathcal{G}$  commutes with all the  $U_3$  generators; the lowering ones can be used successively to generate the full irreducible  $U_3$  basis starting from the function of maximum weight in  $U_3$  defined in (144) as  $P_{(h_1, h_2, h_3)}$ . Hence, the eigenvalue  $g$  of  $\mathcal{G}$  is independent of the row of the irreducible basis and it suffices to find  $g$  in the equation

$$\mathcal{G} P_{(h_1, h_2, h_3)} = g P_{(h_1, h_2, h_3)}.$$

But, expanding  $\mathcal{Y}$  one has

$$\begin{aligned}\mathcal{Y} &= \sum_{q, q'} C_q^{q'} C_{q'}^q = \sum_q (C_q^q)^2 + \sum_{q < q'} C_q^{q'} C_{q'}^q + \sum_{q > q'} C_q^{q'} C_{q'}^q \\ &= \sum_q (C_q^q)^2 + \sum_{q < q'} [C_q^{q'}, C_{q'}^q] + 2 \sum_{q' < q} C_q^{q'} C_{q'}^q\end{aligned}$$

where the effect of the last term, involving a raising operator on the right, vanishes because of (144b) and the second term upon use of relations (135) gives

$$\begin{aligned}\mathcal{Y}P &= \left\{ \sum_q (C_q^q)^2 + \sum_{q < q'} (C_q^q - C_{q'}^{q'}) \right\} P \\ &= \{ C_1^1 + C_2^2 + C_3^3 + 2(C_1^1 - C_3^3) \} P \quad (174)\end{aligned}$$

$$\therefore g = h_{13}^2 + h_{23}^2 + h_{33}^2 + 2(h_{13} - h_{33}) \quad (175)$$

depending only on the  $U_3$  representation  $(h_{13}, h_{23}, h_{33})$ . This representation can alternatively be characterized by only two numbers  $(k_1, k_2) \equiv (h_{13} - h_{33}, h_{23} - h_{33})$  since (172) gives

$h_{13} + h_{23} + h_{33} = \nu N = \text{constant for a given problem.}$  (The relation between  $(k_1, k_2)$  and the labels  $(\lambda, \mu)$  used by Elliott in his work<sup>12)</sup> are  $\lambda = k_1 - k_2$  and  $\mu = k_2$ .) Hence, the eigenvalue of  $Q^2$  is

$$E_L^{(k_1, k_2)} = g - \frac{1}{2} L(L+1) - \frac{1}{3} (vN)^2$$

$$E_L^{(k_1, k_2)} = \frac{2}{3} (k_1 + k_2)^2 - 2 k_1 (k_2 - 1) - \frac{1}{2} L(L+1) \quad (176)$$

manifesting a rotational band-like structure due to the  $L(L+1)$  term. States (145c) are eigenfunctions of  $Q^2$ :

$$Q^2 | [f] \alpha(k_1, k_2) \omega L M_L \rangle = E_L^{(k_1, k_2)} | [f] \alpha(k_1, k_2) \omega L M_L \rangle \quad (177)$$

## 2. Exchange Dependent On Spin and Isospin. The Group $U_4$ .<sup>87)</sup>

We shall now include the effect of exchange forces of the Wigner, Majorana, Bartlett and Heisenberg types for the particularly simple extreme of long range. It will be recalled from (147) that a central two-body interaction with exchange can be taken as

$$\sum_{i < j} I_{ij} V(r_{ij}) \equiv \sum_{i < j} (W + M P_{ij}^r - H P_{ij}^{\tau} + B P_{ij}^{\sigma}) V(r_{ij}). \quad (178a)$$

The operator of exchange between the  $i^{th}$  and  $j^{th}$  particles is  $I_{ij}$  (for interchange) and is defined as the linear combination within the parenthesis of (178a) of the operators

$$\text{Bartlett: } P_{ij}^{\sigma} = \frac{1}{2} (1 + \vec{\sigma}_i \cdot \vec{\sigma}_j) = \frac{1}{2} (1 + 4 \vec{S}_i \cdot \vec{S}_j)$$

Heisenberg:  $P_{ij}^T = \frac{1}{2} (1 + \vec{\tau}_i \cdot \vec{\tau}_j) = \frac{1}{2} (1 + 4 \vec{t}_i \cdot \vec{t}_j) \quad (178b,c,d)$

Majorana:  $P_{ij}^r = -\frac{1}{4} (1 + 4 \vec{\tau}_i \cdot \vec{\tau}_j) (1 + 4 \vec{t}_i \cdot \vec{t}_j)$

in accordance with (148),  $\vec{s}$  and  $\vec{t}$  being the spin and isospin operators of a single particle with possible eigenvalues each of 1/2 and -1/2.

Now, by formula (34) our two-body operator (178a) in second-quantization language will become

$$\mathcal{V} = \frac{1}{2} \sum_{\substack{p_1 p_2 \\ p'_1 p'_2}} \langle p_1 p_2 | I_{12} V_{12} | p'_1 p'_2 \rangle \{ C_{p_1}^{p'_1} C_{p_2}^{p'_2} - \delta_{p_2}^{p'_1} C_{p_1}^{p'_2} \}$$

where  $p_i \equiv (\mu_i, \sigma_i, \tau_i)$ ,  $(i=1,2)$ . Since  $I_{12}$  depends only on spin-isospin and  $V_{12}$  only on the orbital part, then

$$\mathcal{V} = \frac{1}{2} \sum_{\substack{\mu_1 \sigma_1 \tau_1 \\ \mu_2 \sigma_2 \tau_2 \\ \mu'_1 \sigma'_1 \tau'_1 \\ \mu'_2 \sigma'_2 \tau'_2}} \langle \sigma_1 \tau_1, \sigma_2 \tau_2 | I_{12} | \sigma'_1 \tau'_1, \sigma'_2 \tau'_2 \rangle \langle \mu_1 \mu_2 | V_{12} | \mu'_1 \mu'_2 \rangle \times \\ \times \left\{ C_{\mu_1 \sigma_1 \tau_1}^{\mu'_1 \sigma'_1 \tau'_1} C_{\mu_2 \sigma_2 \tau_2}^{\mu'_2 \sigma'_2 \tau'_2} - \delta_{\mu_2}^{\mu'_1} \delta_{\sigma_2}^{\sigma'_1} \delta_{\tau_2}^{\tau'_1} C_{\mu_1 \sigma_1 \tau_1}^{\mu'_2 \sigma'_2 \tau'_2} \right\}$$

Assuming  $V(r_{12}) \equiv V_{12}$  equal to say  $-V_0 e^{-r_{12}^2/\alpha^2}$  to be of long range then we can approximate

$$V_{12} \simeq -V_0 \quad (\text{constant})$$

so that  $\langle \mu_1 \mu_2 | V_{12} | \mu'_1 \mu'_2 \rangle \simeq -V_0 \delta_{\mu_1}^{\mu'_1} \delta_{\mu_2}^{\mu'_2}$  and thus, since in this case  $\mathcal{V} = -V_0$ .

$$J = \frac{1}{2} \sum_{\substack{\sigma_1 \tau_1, \sigma_2 \tau_2 \\ \sigma'_1 \tau'_1, \sigma'_2 \tau'_2}} \langle \sigma_1 \tau_1, \sigma_2 \tau_2 | I_{12} | \sigma'_1 \tau'_1, \sigma'_2 \tau'_2 \rangle \times \quad (179)$$

$$\times \left\{ C_{\sigma_1 \tau_1}^{\sigma'_1 \tau'_1} C_{\sigma_2 \tau_2}^{\sigma'_2 \tau'_2} - \delta_{\sigma_2}^{\sigma'_2} \delta_{\tau_2}^{\tau'_2} C_{\sigma_1 \tau_1}^{\sigma'_1 \tau'_1} \right\}$$

where from (61) the  $4^2$  operators  $C_{\sigma \tau}^{\sigma' \tau'}$ , which obey the  $U_4$  commutation relations (62), are the generators of  $U_4$  --- the group of transformations in spin-isospin space.

The problem of the complete classification of states of supermultiplet theory, i.e., according to the  $U_4$  chain given in (99), has been solved recently by Moshinsky and Nagel.<sup>46)</sup> To completely characterize the rows of the bases for the irreducible representations of  $U_n$ , Racah in his Lecture Notes<sup>55)</sup> proves that  $\frac{1}{2}n(n-1)$  commuting operators that are functions of the group generators are required. This, for  $U_4$  six such operators are needed. Explicit reduction of  $U_4$  representations according to (99) provides four of the six needed:  $S^2$ ,  $T^2$ ,  $S_z$ ,  $T_z$  with eigenvalues  $S(S+1)$ ,  $T(T+1)$   $M_S$  and  $M_T$ . (The other two, whose eigenvalues constitute the additional label  $\beta$  in the chain (99), are derived by Moshinsky & Nagel in the paper referred to above.) The cited authors redefine the  $U_4$  group generators as the following more physical operators, totaling  $4^2$  in number,

$$\mathcal{U} \equiv \sum_{\sigma\tau\sigma'\tau'} (M_0)_{\sigma'}^{\sigma} (N_0)_{\tau'}^{\tau} C_{\sigma\tau}^{\sigma'\tau'} = \sum_{\sigma\tau} C_{\sigma\tau}^{\sigma\tau}$$

$$S_k \equiv \frac{1}{2} \sum_{\sigma\tau\sigma'\tau'} (M_k)_{\sigma'}^{\sigma} (N_0)_{\tau'}^{\tau} C_{\sigma\tau}^{\sigma'\tau'}$$

(180 a, b, c, d)

$$T_l \equiv \frac{1}{2} \sum_{\sigma\tau\sigma'\tau'} (M_0)_{\sigma'}^{\sigma} (N_l)_{\tau'}^{\tau} C_{\sigma\tau}^{\sigma'\tau'}$$

$$R_{kl} \equiv \frac{1}{4} \sum_{\sigma\tau\sigma'\tau'} (M_k)_{\sigma'}^{\sigma} (N_l)_{\tau'}^{\tau} C_{\sigma\tau}^{\sigma'\tau'}$$

where  $k, l=1,2,3$  refer to cartesian components and

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the unit matrix and the well-known Pauli spin-matrices whose rows and columns are given by  $\sigma', \sigma = 1/2, -1/2$ . Similarly,  $N_0, N_l$  ( $l=1,2,3$ ) refer to the unit and Pauli matrices in isospin with rows and columns given by  $\tau', \tau = 1/2, -1/2$ . Among the commutation relations between the operators (180), the relations

$$[S_i, S_j] = i \sum_k \epsilon_{ijk} S_k; \quad [T_i, T_j] = i \sum_k \epsilon_{ijk} T_k; \quad [S_i, T_j] = 0$$

hold ----- thus  $S_k$  ( $k=1,2,3$ ) and  $T_l$  ( $l=1,2,3$ ) are the generators of  $SU_2$ (spin) and  $SU_2$ (isospin) which in turn are homomorphic to



$R_3$  (spin) and  $F_3$  (isopin), respectively. Now, a tensor operator  $T_{qr}$  (similar to (154) for  $U_3$ ) can here be defined as

$$T_{qr} \equiv \sum_{\sigma\tau\sigma'\tau'} (M_q)_\sigma^\sigma (N_r)_\tau^{\tau'} C_{\sigma\tau}^{\sigma'\tau'} \quad \begin{cases} q=0, k=0,1,2,3 \\ r=0, l=0,1,2,3 \end{cases} \quad (181)$$

such that

$$T_{00} = W, \quad T_{k0} = 2S_k, \quad T_{0l} = 2T_l, \quad T_{kl} = 4R_{kl},$$

and whose scalar product

$$\begin{aligned} \sum_{qr} T_{qr} T_{qr} &= T_{00}^2 + \sum_k T_{k0}^2 + \sum_l T_{0l}^2 + \sum_{kl} T_{kl}^2 \\ &= W^2 + 4S^2 + 4T^2 + 16R^2. \end{aligned} \quad (182)$$

On the other hand, from definition (181)

$$\begin{aligned} \sum_{qr} T_{qr} T_{qr} &= \sum_{\substack{\sigma\tau\sigma'\tau' \\ \bar{\sigma}\bar{\tau}\bar{\sigma}'\bar{\tau}'}} \sum_q (M_q)_\sigma^\sigma (M_q)_{\bar{\sigma}}^{\bar{\sigma}'} \sum_r (N_r)_\tau^{\tau'} (N_r)_{\bar{\tau}}^{\bar{\tau}'} C_{\sigma\tau}^{\sigma'\tau'} C_{\bar{\sigma}\bar{\tau}}^{\bar{\sigma}'\bar{\tau}'} \\ &= \sum_{\substack{\sigma\tau\sigma'\tau' \\ \bar{\sigma}\bar{\tau}\bar{\sigma}'\bar{\tau}'}} 2\delta_{\sigma'}^{\bar{\sigma}'} \delta_{\bar{\sigma}}^{\sigma} \cdot 2\delta_{\tau'}^{\bar{\tau}'} \delta_{\bar{\tau}}^{\tau} C_{\sigma\tau}^{\sigma'\tau'} C_{\bar{\sigma}\bar{\tau}}^{\bar{\sigma}'\bar{\tau}'} \\ &= 4 \sum_{\substack{\sigma\tau \\ \sigma'\tau'}} C_{\sigma\tau}^{\sigma'\tau'} C_{\sigma'\tau'}^{\sigma\tau} \\ &= 4 G_2(U_4) \end{aligned} \quad (183)$$

where  $G_2(U_4)$  is the (second degree) Casimir operator of  $U_4$ .

Therefore,

$$G_2(U_4) = \frac{1}{4} W^2 + S^2 + T^2 + 4R^2. \quad (184)$$

So much for the formal properties of the group  $U_4$ .

Turning our attention to the exchange operator  $I$  of (179), and in view of definition (178a) for  $I_{ij}$ , one concludes that

$$I = W + M P^r + B P^s - H P^t \quad (185)$$

$W, M, B$  and  $H$  being arbitrary constants as before; the script  $P$  referring to the second-quantization formulation of the various exchange operators. For example, the isopin independent Bartlett operator is now

$$P^s = \frac{1}{2} \sum_{\substack{\sigma_1, \sigma_2 \\ \sigma'_1, \sigma'_2}} \langle \sigma_1, \sigma_2 | P_{12}^s | \sigma'_1, \sigma'_2 \rangle \sum_{\tau_1, \tau_2} \{ C_{\sigma_1 \tau_1}^{\sigma'_1 \tau_1} C_{\sigma_2 \tau_2}^{\sigma'_2 \tau_2} - \delta_{\sigma_2}^{\sigma'_1} \delta_{\tau_2}^{\tau_1} C_{\sigma_1 \tau_1}^{\sigma'_2 \tau_2} \}.$$

But from (178b), the coefficient

$$\frac{1}{2} \langle \sigma_1, \sigma_2 | P_{12}^s | \sigma'_1, \sigma'_2 \rangle = \frac{1}{4} \delta_{\sigma_1}^{\sigma'_1} \delta_{\sigma_2}^{\sigma'_2} + \sum_{k=1}^3 \langle \sigma_1 | \Delta_{k1} | \sigma'_1 \rangle \langle \sigma_2 | \Delta_{k2} | \sigma'_2 \rangle$$

so that from (180a,b) the operator  $P^s$  becomes simply

$$P^s = \frac{1}{4} W^2 - W + S^2$$

if one recalls that, for example  $\langle \sigma | \Delta_k | \sigma' \rangle$  is one-half the

( $\sigma, \sigma'$ )-element of the  $k^{\text{th}}$  component Pauli spin matrix. In exactly the same manner, the spin-independent Heisenberg operator reduces to

$$P^{\tau} = \frac{1}{2} W^2 - W + T^2$$

The Majorana operator (178d) is more general than these; it can be written as

$$\begin{aligned} P_{12}^r &= -\frac{1}{4} - \vec{\sigma}_1 \cdot \vec{\sigma}_2 - \vec{t}_1 \cdot \vec{t}_2 - 4(\vec{\sigma}_1 \cdot \vec{\sigma}_2)(\vec{t}_1 \cdot \vec{t}_2) \\ &= -\frac{1}{2}(P_{12}^{\sigma} + P_{12}^{\tau}) + \frac{1}{4} - 4 \sum_{k,l=1}^3 \sigma_{k1} t_{l1} \sigma_{k2} t_{l2} . \end{aligned}$$

Thus, using formula (179) one has

$$\begin{aligned} P^r &= -\frac{1}{2}(P^{\sigma} + P^{\tau}) + \frac{1}{2} \left\{ \frac{1}{4}(W^2 - W) \right. \\ &\quad - 4 \sum_{kl} \sum_{\substack{\sigma_1 \tau_1 \\ \sigma'_1 \tau'_1}} \langle \sigma_1 | \sigma_{k1} | \sigma'_1 \rangle \langle \tau_1 | t_{l1} | \tau'_1 \rangle C_{\sigma_1 \tau_1}^{\sigma'_1 \tau'_1} \times \\ &\quad \times \sum_{\substack{\sigma_2 \tau_2 \\ \sigma'_2 \tau'_2}} \langle \sigma_2 | \sigma_{k2} | \sigma'_2 \rangle \langle \tau_2 | t_{l2} | \tau'_2 \rangle C_{\sigma_2 \tau_2}^{\sigma'_2 \tau'_2} \\ &\quad \left. + 4 \sum_{kl} \sum_{\substack{\sigma_1 \tau_1 \sigma_2 \tau_2 \\ \sigma'_1 \tau'_1 \sigma'_2 \tau'_2}} \langle \sigma_1 | \sigma_{k1} | \sigma'_1 \rangle \langle \sigma_2 | \sigma_{k2} | \sigma'_2 \rangle \langle \tau_1 | t_{l1} | \tau'_1 \rangle \langle \tau_2 | t_{l2} | \tau'_2 \rangle \times \right. \\ &\quad \left. \times C_{\sigma_1 \tau_1}^{\sigma'_1 \tau'_1} \delta_{\sigma_2}^{\sigma'_2} \delta_{\tau_2}^{\tau'_2} \right\} \end{aligned}$$

$$= -\frac{1}{2} (P^\sigma + P^\tau) + \frac{1}{2} \left\{ \frac{1}{4} (W^2 - W') - 4R^2 + \frac{9}{4} W' \right\}$$

where the scalar product of (180d) with itself is used in evaluating the expression before the last and the last expression is simply

$$4 \cdot \frac{3}{4} \cdot \frac{3}{4} W' \text{ so that}$$

$$\begin{aligned} P^r &= -\frac{1}{2} (P^\sigma + P^\tau) + \frac{1}{2} \left\{ \frac{1}{4} (W^2 - W') - 4R^2 + \frac{9}{4} W' \right\} \\ &= 2W - \frac{1}{2} \left[ \frac{1}{4} W^2 + 4R^2 + S^2 + T^2 \right] \end{aligned}$$

$$P^r = 2W - \frac{1}{2} G_2(U_4)$$

having used here the previous  $U_4$  result (184). To deal with  $G_2(U_r)$  instead of  $G_2(U_4)$  ----- future developments will make this more convenient ----- one can derive a relation between these two Casimir operators which is very simply due to the fact that the  $U_4$  representations must be conjugate to those of  $U_r$  (see example in 80). From the last two steps in (183) and from (61) and (31) one can write

$$G_2(U_4) = \sum_{ss'} C_s^{s'} C_{s'}^s = \sum_{ss'} \sum_{\mu\bar{\mu}} b_{\mu s}^+ b^{\mu s'} b_{\bar{\mu} s'}^+ b^{\bar{\mu} s}$$

$$(s, s' = \sigma, \tau, \sigma', \tau') \quad (\mu, \mu' = 1, 2, \dots, r)$$

which by using anticommutation relations (12) becomes

$$G_2(U_4) = (4+r)W - \sum_{\mu\bar{\mu}} b_{\mu}^{\bar{\mu}} b_{\bar{\mu}}^{\mu}$$

$$G_2(U_4) = (4+r)W - G_2(U_r)$$

the latter being the  $U_r$  Casimir operator. Thus

$$P^r = \frac{1}{2} [G_2(U_r) - r W].$$

Combining the results just obtained for  $P, P^r$  and  $P^r$  into (185) we arrive finally at the long-range exchange operator

$$\mathcal{J} = W + \frac{1}{4}(B-H)W(W-4) + BS^2 - HT^2 + \frac{1}{2}M[G_2(U_r) - rW]$$

which is an eigenoperator of a basis set transforming irreducibly according to the  $U_4 \supset SU_2 \times SU_2$  segment of the chain in (99), that is, when acting on the kets

$$|\{v_1 v_2 v_3 v_4\} \beta S M_S T M_T\rangle \quad (185)$$

where  $\{v_1 v_2 v_3 v_4\} = [\widetilde{h_1 h_2 \dots h_r}] = [\tilde{f}]$  of (97). The Casimir operator  $G_2(U_r)$  is the operator  $\Gamma^r$  to be used in the section on pairing and whose eigenvalue is there derived to give (226).

Finally, one has the algebraic eigenvalue equation

$$\mathcal{Q} | [\tilde{f}] \beta S M_S T M_T \rangle = I | [\tilde{f}] \beta S M_S T M_T \rangle \quad (186)$$

$$I \equiv W + \frac{1}{4} (B-H) N(N-4) + B S(S+1) - H T(T+1) + \frac{1}{2} M \sum_{\mu=1}^r h_{\mu} (h_{\mu} - 2\mu + 1)$$

where  $I$  is the contribution to the energy at long-range due to exchange.

Conclusion: the simple eigenvalue  $I$  given above depends only on  $S$  and  $T$  for a given  $[f]$  partition of  $N$  particles. One must multiply the diagonal  $\mathcal{Q}^2$  matrix elements of (177) in a base extended to include total spin  $S$  and isospin  $T$  quantum numbers. Thus

$$\begin{aligned} & \langle [f] \alpha(k_1 k_2) \omega L M_L; [\tilde{f}] \beta S M_S T M_T | \mathcal{Q}^2 | [f] \alpha(k_1 k_2) \omega L M_L; [\tilde{f}] \beta S M_S T M_T \rangle \\ &= \langle [f] \alpha(k_1 k_2) \omega L M_L | \mathcal{Q}^2 | [f] \alpha(k_1 k_2) \omega L M_L \rangle \times \\ & \quad \times \langle [\tilde{f}] \beta S M_S T M_T | \mathcal{Q} | [\tilde{f}] \beta S M_S T M_T \rangle \end{aligned} \quad (187)$$

is diagonal and accounts for the long-range part of a central interaction with exchange.

One final remark regarding different exchange mixtures is in order; for this end we give several of the more popular ones in Table IV.2.1.

MIXTURE NAME	W	M	H	B
ROSENFELD	-.13	.93	-.26	.46
SERBER	.50	.50	0	0
KURATH	0	.80	0	.20
MESHKOV	.40	.40	0	.20
SOPER	.40	.30	.10	.20
SLIV <sup>56)</sup>	.70	.15	.15	0

TABLE IV.2.1

Note:  $W+M+H+B \equiv 1$

### 3. The Pairing Force. Short-Range Character.

The general interaction operator  $\sum_{i < j} V(r_{ij})$  is equivalent under second-quantization formulation to the operator (25):

$$\mathcal{V} = \frac{1}{2} \sum_{\substack{p_1, p_2 \\ p'_1, p'_2}} \langle p_1, p_2 | V_{12} | p'_1, p'_2 \rangle b_{p_2}^+ b_{p_1}^+ b_{p'_1} b_{p'_2}, \quad (188)$$

$$p_i \equiv (\nu l_i m_i; \sigma_i \tau_i) \equiv (\nu l_i m_i; s_i) \quad (i=1,2)$$

being the single-particle state labels referring to orbital quantum numbers  $\nu l_i m_i$  arising within a single given shell  $\nu$  and spin-isospin quantum numbers  $S_i$ . As before, if  $V_{12} \equiv V(r_{12})$  is spin-isospin independent then

$$\begin{aligned} \langle \nu l_1 m_1, \nu l_2 m_2 | V_{12} | \nu l'_1 m'_1, \nu l'_2 m'_2 \rangle &= \langle \nu l_1 m_1, \nu l_2 m_2 | V_{12} | \nu l'_1 m'_1, \nu l'_2 m'_2 \rangle \delta_{S_1 S'_1} \delta_{S_2 S'_2} \\ &= \langle \nu l_1 m_1, \nu l_2 m_2 | V_{12} | \nu l'_1 m'_1, \nu l'_2 m'_2 \rangle \delta_{S_1 S'_1} \delta_{S_2 S'_2} \end{aligned} \quad (189)$$

holds. Furthermore, letting  $\Lambda$  be the resultant orbital angular momentum eigenvalue and  $M$  its projection, one has the coupled or paired state

$$|\nu l_1, \nu l_2, \Lambda M\rangle = \sum_{m_1, m_2} \langle l_1, l_2, m_1, m_2 | \Lambda M \rangle |\nu l_1 m_1, \nu l_2 m_2\rangle, \quad (190a)$$

the Clebsch-Gordan coefficients obeying the usual orthonormality relations

$$\sum_{m_1, m_2} \langle l_1, l_2, m_1, m_2 | \Lambda M \rangle \langle l_1, l_2, m_1, m_2 | \Lambda' M' \rangle = \delta_{\Lambda \Lambda'} \delta_{M M'}. \quad (190b)$$

Let us rewrite (188) such that particles 1 and 2 are paired to give net angular momentum  $\Lambda$ . Obviously

$$\begin{aligned} \mathcal{V} = \frac{1}{2} \sum_{\substack{l_1, l_2, l'_1, l'_2 \\ m_1, m_2, m'_1, m'_2 \\ \bar{m}_1, \bar{m}_2, \bar{m}'_1, \bar{m}'_2}} \langle \nu l_1 \bar{m}_1, \nu l_2 \bar{m}_2 | V_{12} | \nu l'_1 \bar{m}'_1, \nu l'_2 \bar{m}'_2 \rangle \delta_{\bar{m}_1 \bar{m}'_1} \delta_{\bar{m}_2 \bar{m}'_2} \delta_{\bar{m}'_1 \bar{m}'_2} \delta_{\bar{m}_1 \bar{m}_2} \\ \times \sum_{S_1, S_2} b_{\nu l_2 m_2 S_2}^+ b_{\nu l_1 m_1 S_1}^+ b_{\nu l'_1 m'_1 S_1} b_{\nu l'_2 m'_2 S_2} \end{aligned}$$



is equivalent to (188) with restriction (189). Using (190) and (191) there results

$$\mathcal{V} = \frac{1}{2} \sum_{\substack{\Lambda \Lambda' M M' \\ l, l_2 l'_2}} \langle \nu l_1, \nu l_2, \Lambda M | V_{12} | \nu l'_1, \nu l'_2, \Lambda' M' \rangle \delta_{\Lambda \Lambda'} \delta_{M M'} \\ \times \sum_{S, S_2} \left\{ \sum_{m_1 m_2} \langle l, l_2 m_1 m_2 | \Lambda M \rangle b_{\nu l_2 m_2 S_2}^+ b_{\nu l_1 m_1 S_1}^+ \right\} \left\{ \sum_{m'_1 m'_2} \langle l'_1 l'_2 m'_1 m'_2 | \Lambda' M' \rangle b_{\nu l'_1 m'_1 S_1} b_{\nu l'_2 m'_2 S_2} \right\}$$

the  $\delta_{\Lambda \Lambda'} \delta_{M M'}$  appearing because of spherical (and of course, also axial) symmetry, i.e.,  $L^2$  and  $L_z$  commute with  $V_{12}$ . The coefficient matrix element, moreover, is independent of the row  $M = \Lambda, \Lambda - 1, \dots, -\Lambda$  (see Appendix F) so that suppressing the redundant label  $M$  in that coefficient, one has that

$$\mathcal{V} = \frac{1}{2} \sum \langle \nu l_1, \nu l_2, \Lambda | V_{12} | \nu l'_1, \nu l'_2, \Lambda \rangle P(l'_1 l'_2, l_1 l_2, \Lambda) \quad (191a)$$

where

$$P(l'_1 l'_2, l_1 l_2, \Lambda) \equiv \sum_{S, S_2, M} \left\{ \sum_{m_1 m_2} \langle l, l_2 m_1 m_2 | \Lambda M \rangle b_{\nu l_2 m_2 S_2}^+ b_{\nu l_1 m_1 S_1}^+ \right\} \quad (191b) \\ \times \left\{ \sum_{m'_1 m'_2} \langle l'_1 l'_2 m'_1 m'_2 | \Lambda M \rangle b_{\nu l'_1 m'_1 S_1} b_{\nu l'_2 m'_2 S_2} \right\}$$

can be considered a "generalized pairing force"<sup>57)</sup> ----- it acts on a many-particle state of the type (21) destroying a particle-pair in the orbitals  $l'_1 l'_2$  and of total angular momentum  $\Lambda$  and then creates another pair in the orbitals  $l_1 l_2$  having the same resultant  $\Lambda$ .

A pairing force with  $\lambda=0$  considered in the case of superconductivity<sup>58,59)</sup> directly predicts the experimental energy gap between Fermi surface and conduction band. A similar interaction between nucleons was considered by Bohr, Mottelson & Pines<sup>14)</sup> which among other things accounts for the well-defined large energy spacing between the ground and excited levels in even-even nuclei<sup>14, 15)</sup>. As early as 1943 Racah<sup>60)</sup> calculated interaction matrix elements of a pairing force of this type in some complex atoms using the expression

$$\langle l^2, \lambda | V_{12} | l^2, \lambda \rangle = (2l+1) \delta_{\lambda 0}$$

for a pair of "equivalent" electrons in an orbital state  $l$ . For non-equivalent ones (mixed orbitals  $ll'$ ) this can be generalized to

$$\langle ll, \lambda | V_{12} | l'l', \lambda \rangle = \sqrt{(2l+1)(2l'+1)} \delta_{\lambda 0}.$$

Inserting this value for the coefficients in (191) one obtains Moshinsky's form of the pairing operator

$$\begin{aligned} P &\equiv \sum_{ll'} \sqrt{(2l+1)(2l'+1)} P(l'l', ll, 0) \\ &= \frac{1}{2} \sum_{\substack{l, m \\ l', m'}} \sum_{s, s_2} (-)^{m+m'} b_{vl-m, s_2}^+ b_{vl, m, s}^+ b_{vl', m', s_1} b_{vl', -m', s_2} \end{aligned}$$

where use is made of the well-known result that

$$\langle ll, m-m | 00 \rangle = \frac{(-)^{l-m}}{\sqrt{2l+1}}.$$

The operator  $P$  clearly displays the required pairing property. A

more convenient form in terms of  $U_r$  group generators follows from anticommutation relations (12) and definitions (31) and (37):

$$P = \frac{1}{2} \sum_{\substack{l m \\ l' m'}} (-)^{m+m'} b_{vlm}^{vl'm'} b_{vl-m}^{vl'-m'} - \frac{1}{2} N \quad (192)$$

$$N \equiv \sum_{l m} b_{vlm}^{vlm}$$

$$(l = v, v-2, \dots, 1 \text{ or } 0) \quad (m = l, l-1, \dots, -l)$$

where  $N$  is the particle-number operator defined previously.

### 3a. Single-Orbital Pairing and the Group $R_{2l+1}$

The special case of pairing between nucleons in a single orbital state  $l$  is given by the operator

$$P = \frac{1}{2} \sum_{m m'} (-)^{m+m'} b_m^{m'} b_{-m}^{-m'} - \frac{1}{2} N \quad (193)$$

$$N = \sum_{m=-l}^l b_m^m$$

which follows immediately from (192) if  $l \equiv l'$ . The set of  $(2l+1)^2$  operators

$$L_m^{m'} = \sum_s b_{lms}^+ b^{lms} \quad (194)$$

are the generators of  $U_{2l+1}$ . The  $(2l+1)$ -dimensional space is spanned by the  $(2l+1)$  vector components  $b_{lms}^+$  which under rotations in this space transform like the spherical harmonic components  $Y_{lm}(\theta, \varphi)$ . The scalar product

$$\sum_{mm'} g_{mm'} b_{ms}^+ b^{m's'} = \sum_m (-)^m b_{ms}^+ b_{-ms'}^+$$

must remain invariant under  $(2l+1)$ -dimensional rotations. The space metric  $g_{mm'}$  is thus seen to be

$$g_{mm'} = (-)^m \delta_{m, -m'} \quad (195)$$

which also gives the rule for raising and lowering indices since, for instance,

$$L_{mm'} = \sum_{m''} g_{m'm''} L_m^{m''} = (-)^{m'} L_m^{-m'} \quad (196)$$

The  $U_{2l+1}$  generators, in doubly-covariant form, thus obey the Lie algebra

$$[L_{mm'}, L_{m''m'''}] = g_{m'm''} L_{mm'''} - g_{m'''m} L_{m''m'}.$$

Defining the antisymmetric operator set<sup>33)</sup>

$$\Lambda_{mm'} \equiv \frac{1}{2} (\ell_{mm'} - \ell_{m'm}), \quad \Lambda_{mm'} = -\Lambda_{m'm} \quad (197a, b)$$

one can see that these obey the commutation relations

$$[\Lambda_{mm'}, \Lambda_{m''m''}] = \quad (198)$$

$$\frac{1}{2} \{ g_{m''m'} \Lambda_{mm''} + g_{m''m} \Lambda_{m'm''} + g_{mm''} \Lambda_{m''m'} + g_{m'm''} \Lambda_{m''m} \}$$

which from Appendix E are seen to be the generators for infinitesimal rotations in  $(2l+1)$ -dimensions, i.e., of the group  $R_{2l+1}$ . From (197b) it follows that there are  $l(2l+1)$  linearly independent generators of this kind. The operator set  $\{\Lambda_{mm'}\}$  is contained in the set  $\{\ell_m^{m'}\}$  of  $U_{2l+1}$ , so that  $U_{2l+1} \supset R_{2l+1}$  as a subgroup. Using (196) we can express the set (197) more conveniently as

$$\Lambda_m^{m'} = \frac{1}{2} (\ell_m^{m'} + (-)^{m+m'+1} \ell_{-m'}^{-m}), \quad \Lambda_m^{m'} = (-)^{m+m'+1} \Lambda_{-m'}^{-m} \quad (199)$$

the latter symmetry rule again confirming that one has  $l(2l+1)$  independent generators. From (38) can be derived the relations

$$[\Lambda_m^{m'}, \Lambda_m^{m''}] = \quad (200)$$

$$\frac{1}{2} \{ \Lambda_m^{m'''} \delta_m^{m'} - \Lambda_m^{m'} \delta_m^{m''} + (-)^{m+m'+1} \Lambda_{-m'}^{m'''} \delta_{m''}^{-m} + (-)^{m+m'} \Lambda_{-m''}^{-m} \delta_{-m'}^{m'''} \}$$

equivalent to the algebra (198) but now in a form more convenient for us as will be seen presently.

The contraction of  $\mathcal{U}_{2l+1}$  generators

$$\Gamma \equiv \sum_{m m'} \ell_m^{m'} \ell_{m'}^m \quad (201)$$

is seen to be a  $\mathcal{U}_{2l+1}$  invariant since

$$[\Gamma, \ell_m^{m'}] = 0 \quad (202)$$

The equivalent contraction over  $R_{2l+1}$  generators

$$\Phi \equiv \sum_{m m'} \Lambda_m^{m'} \Lambda_{m'}^m \quad (203)$$

is an  $R_{2l+1}$  invariant as

$$[\Phi, \Lambda_m^{m'}] = 0 \quad (204)$$

Thus,  $\Gamma$  and  $\Phi$  satisfy the first two Racah conditions<sup>61)</sup> for Casimir operators (given on page 30 in relation to the group  $R_3$ ).

Introducing a numbering convention  $\mu = l - m + 1$  (integer) such that there is a one-to-one correspondence between  $m = l, l-1, \dots, -l$

and  $\mu = 1, 2, \dots, 2l+1$  we have, first of all,

$$\Gamma = \sum_{\mu, \mu'=1}^{2l+1} b_{\mu}^{\mu'} b_{\mu'}^{\mu} = \sum_{\mu} (b_{\mu}^{\mu})^2 + \sum_{\mu < \mu'} [b_{\mu}^{\mu'}, b_{\mu'}^{\mu}] + 2 \sum_{\mu < \mu'} b_{\mu'}^{\mu} b_{\mu}^{\mu'}$$

where, from (73b), the effect of the last term when acting on a maximum weight state  $\mathbb{P}$  is zero. The equation

$$\Gamma \mathbb{P} = \gamma \mathbb{P}$$

will have an eigenvalue  $\gamma$  (valid for any row of the  $\mathcal{U}_{2l+1}$  irreducible basis, since  $\Gamma$  is a group invariant) that is evaluated identically as in (173-175) to give the result:

$$\gamma = \sum_{\mu=1}^r [h_{\mu} (h_{\mu} - 2\mu + r + 1)] \quad (r=2l+1). \quad (205)$$

Regarding the group  $R_{2l+1}$ , the  $l$  linearly independent (hermitean) operators, which from (200) mutually commute, can be used to define weight in  $R_{2l+1}$  by

$$\Lambda_m^m \bar{P} = \bar{\mu}_m \bar{P} \quad (\mu = l-m+1) \quad (206)$$

$\bar{P}$  being a polynomial of weight  $(\bar{\mu}_m)$ . The set of operators  $\{\Lambda_m^m\}$  is divisible as before into three classes

$$1) \Lambda_m^m (m > m' > -m) \quad 2) \Lambda_m^m (m=l, l-1, \dots, -l) \quad 3) \Lambda_m^m (m > m' > -m) \quad (207)$$

of which class 2 are the weight operators. Let two new polynomials  $\bar{P}'$  and  $\bar{P}''$  be defined as

$$\left. \begin{aligned} \bar{P}' &\equiv \Lambda_{m''}^{m'''} \bar{P} \\ \bar{P}'' &\equiv \Lambda_{m'''}^{m''} \bar{P} \end{aligned} \right\} (m'' > m''' > -m'') \quad (208)$$

Then,

$$\begin{aligned} \Lambda_m^m \bar{P}' &= [\Lambda_m^m, \Lambda_{m''}^{m'''}] \bar{P} + \Lambda_{m''}^{m'''} \Lambda_m^m \bar{P} \\ &= \frac{1}{2} (\Lambda_m^{m'''} \delta_{m''}^m - \Lambda_{m''}^m \delta_m^{m''} - \Lambda_{-m}^{m'''} \delta_{m''}^{-m} + \Lambda_{m''}^{-m} \delta_m^{m''}) \bar{P} \\ &\quad + \Lambda_{m''}^{m'''} \Lambda_m^m \bar{P} \\ &= \left[ \frac{1}{2} \delta_{m''}^m - \frac{1}{2} \delta_m^{m''} - \frac{1}{2} \delta_{m''}^{-m} + \frac{1}{2} \delta_{-m}^{m''} + \bar{u}_\mu \right] \bar{P}' \end{aligned}$$

shows that class 1 consists of raising generators and a similar analysis of  $\bar{P}'$  proves class 3 to be the lowering generators. Hence, maximum weight polynomials  $\bar{P}$  in  $R_{2\ell+1}$  will satisfy

$$\Lambda_m^m \bar{P} = \lambda_\mu \bar{P} \quad \Lambda_{m'}^{m'} \bar{P} = 0 \quad (m > m' > -m) \quad (209a, b)$$

with maximum weight being given by the set of  $K$  numbers

$$(\lambda_1, \lambda_2, \dots, \lambda_K) \quad \left\{ \begin{array}{l} K = \frac{1}{2}(r-1) \\ r = 2\ell+1 \end{array} \right. \quad (210)$$

In a way identical with (75) one can deduce that

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_K \geq 0. \quad (211)$$



Finally, since

$$\Phi \bar{P} = \varphi \bar{P}$$

where  $\varphi$  is the Casimir eigenvalue of  $R_{2l+1}$  which, after a short derivation along the lines arriving at (205), results explicitly in<sup>33)</sup>

$$\varphi = \frac{1}{2} \sum_{\mu=1}^K \lambda_{\mu} (\lambda_{\mu} + r - 2\mu) \quad \begin{cases} K = \frac{1}{2}(r-1) \\ r = 2l+1 \end{cases} \quad (212)$$

A special case of (201) was used to great advantage in the oscillator symmetry group  $U_3$  to determine the  $Q^2$  interaction eigenvalues. The single-orbital pairing force (193) can also be expressed as a linear combination of Casimir operators (second degree ones—and all further reference to Casimir operators will entail only this type) since, an expanding (203) using definition (199) we have, putting  $\lambda \equiv m+m'+1$ ,

$$\begin{aligned} \Phi &= \sum_{mm'} \Lambda_m^{m'} \Lambda_{m'}^m = \frac{1}{4} \sum_{mm'} (b_m^{m'} + (-)^{\lambda} b_{-m'}^{-m}) (b_{m'}^m + (-)^{\lambda} b_{-m}^{-m'}) \\ &= \frac{1}{4} \sum_{mm'} b_m^{m'} b_{m'}^m + \frac{1}{4} \sum_{mm'} (-)^{\lambda} b_m^{m'} b_{-m}^{-m'} \\ &\quad + \frac{1}{4} \sum_{mm'} (-)^{\lambda} b_{-m'}^{-m} b_{m'}^m + \frac{1}{4} \sum_{mm'} b_{-m'}^{-m} b_{-m}^{-m'} \end{aligned}$$

which upon change of some dummy indices and signs becomes

$$\Phi = \frac{1}{2} \sum_{mm'} b_m^{m'} b_{m'}^m + \frac{1}{2} \sum_{mm'} (-)^1 b_m^{m'} b_{-m}^{-m'}$$

$$R_{2L+1} : \quad \Phi = \frac{1}{2} \Gamma - P - \frac{1}{2} N \quad (213)$$

from (201) and (193). Therefore,

$$P = \frac{1}{2} \Gamma - \Phi - \frac{1}{2} N \quad (214)$$

will be diagonal in a base transforming irreducibly under

$$U_{2L+1} \supset R_{2L+1}.$$

The resulting states can be classified, in addition, by total orbital angular momentum  $L$ , i.e., with respect to  $R_3$  whose generators  $L_q$  are expressible as the linear combinations

$$L_q = \sum_{mm'} \sqrt{l(l+1)} \langle l 1 m' q | l m \rangle \Lambda_m^{m'} \quad (q = 1, \bar{1}, 0) \quad (215)$$

which, through (200) can be seen to satisfy the Lie algebra

$$[L_q, L_{q'}] = \sum_{q''} (-)^{q''} \epsilon_{q-q''q'} L_{q''} \quad (q, q' = 1, \bar{1}, 0) \quad (216)$$

coinciding with (142b). Thus, single-orbital pairing is diagonal in a scheme irreducible according to the group chain

$$U_{2L+1} \supset R_{2L+1} \supset R_3 \supset R_2 \quad (217)$$

where by  $R_3$  is meant the  $D^{(L)}(R_3)$  representation of dimension  $(2L+1)$ .

The corresponding ket state is denoted by

$$| \Sigma h_1 h_2 \dots h_{2\ell+1} ] \alpha' (\lambda, \lambda_2 \dots \lambda_\ell) \omega' L M_L \rangle \quad (218)$$

where, again,  $\alpha'$  and  $\omega'$  are additional labels that may be needed to distinguish multiplicities, if they occur.

### 3b. Mixed-orbital pairing and the $R_{2\ell+1}$ group.

If the particles paired to total angular momentum  $\Lambda = 0$  have available more than one orbital then  $\ell \neq \ell'$  in (192). In the chapter on harmonic oscillator symmetry we decided to classify our one-particle oscillator states by  $\mu = (\eta, \eta_1, \eta_0)$  instead of  $(\nu \ell m)$ , where the former are "partitions" along spherical "modes"  $(+, -, 0)$  of oscillator quanta  $\nu = \eta_1 + \eta_1 + \eta_0$ . The single-particle orbital state (creation) operator  $b_{\nu \ell m}^+$  transforms like the spherical harmonics  $Y_{\ell m}(\theta, \varphi)$  which in turn have the phase convention

$$Y_{\ell m}^*(\theta, \varphi) = (-)^m Y_{\ell -m}(\theta, \varphi)$$

of Condon & Shortley.<sup>62)</sup> Thus, following this convention

$$(b_{\nu \ell m}^+)^* = (-)^m b_{\nu \ell -m}^+.$$

On the other hand, the state  $b_{\eta, \eta_1, \eta_0}^+ |0\rangle$ , where  $|0\rangle$  is a vacuum (no-particle) state, should transform exactly like the oscillator state  $|\eta, \eta_1, \eta_0\rangle$  of (123) given in terms of boson creation operators acting on a ground state  $|\bar{0}\rangle$ . Now, since

$$(a_{\pm 1}^+)^* = -a_{\mp 1} \quad \text{according to (116), then the aforementioned state}$$

$|\eta, \eta_i, \eta_o\rangle$  has the property

$$|\eta, \eta_i, \eta_o\rangle^* = (-)^{\eta_i + \eta_i} |\eta_i, \eta, \eta_o\rangle$$

and since  $b_{\eta, \eta_i, \eta_o}^+$  transform like  $|\eta, \eta_i, \eta_o\rangle$  we have

$$(b_{\eta, \eta_i, \eta_o}^+)^* = (-)^{\eta_i + \eta_i} b_{\eta_i, \eta, \eta_o}^+.$$

But in (192) are involved the operators

$$b_{\nu l m}^+; \quad (-)^m b_{\nu l -m}^+,$$

and their respective annihilation counterparts. To the complete set  $\{b_{\nu l m}^+\}$  there corresponds the complete set  $\{b_{\eta, \eta_i, \eta_o}^+\}$  ----- there being in each case  $r = \frac{1}{2}(\nu+1)(\nu+2)$  components. Likewise, to the set  $\{(-)^m b_{\nu l -m}^+\}$  there corresponds the set  $\{(-)^{\eta_i + \eta_i} b_{\eta_i, \eta, \eta_o}^+\}$ . The same holds for annihilation operators. Thus with  $\eta, \eta_i, \eta_o$  indices (192) becomes our (working) pairing operator for mixed-orbitals:

$$P = \frac{1}{2} \sum_{\substack{\eta, \eta_i, \eta_o \\ \eta', \eta'_i, \eta'_o}} (-)^{\eta_i + \eta_i + \eta'_i + \eta'_i} b_{\eta, \eta_i, \eta_o}^{\eta', \eta'_i, \eta'_o} b_{\eta_i, \eta, \eta_o}^{\eta'_i, \eta'_i, \eta'_o} - \frac{1}{2} N. \quad (219)$$

The generators defined in (199) now become

---

General formulas and tables for the 2s-1d shell of the transformation coefficients from one scheme to the other have been published by Chacon and the author.<sup>(3)</sup>

$$\Lambda_{n, n_1, n_0}^{n', n'_1, n'_0} = \frac{1}{2} \left[ \mathcal{C}_{n, n_1, n_0}^{n', n'_1, n'_0} + (-)^{n_1 + n_1' + n'_1 + n'_1 + 1} \mathcal{C}_{n_1', n_1', n_0'}^{n_1, n_1, n_0} \right] \quad (220a, b)$$

$$\Lambda_{n, n_1, n_0}^{n', n'_1, n'_0} = (-)^{n_1 + n_1' + n'_1 + n'_1 + 1} \Lambda_{n_1', n_1', n_0'}^{n_1, n_1, n_0}$$

where the former (symmetry) relation shows that one has  $\frac{1}{2} r(r-1)$  linearly independent such operators.

Let us adopt the shorthand notation

$$\begin{aligned} (n, n_1, n_0) &\equiv \mu & (n', n'_1, n'_0) &\equiv \mu' \\ (n_1, n_1', n_0) &\equiv \mu_- & (n_1', n_1, n_0') &\equiv \mu'_- \end{aligned} \quad (121)$$

there being  $r = \frac{1}{2} (V+1)(V+2)$  possibilities for each  $\mu$ . It then becomes simpler to prove that

$$\begin{aligned} [\Lambda_{\mu}^{\mu'}, \Lambda_{\mu''}^{\mu''}] &= \\ \frac{1}{2} \{ \delta_{\mu''}^{\mu'} \Lambda_{\mu}^{\mu''} - \delta_{\mu}^{\mu''} \Lambda_{\mu''}^{\mu'} + (-)^{\eta+1} \delta_{\mu}^{\mu''} \Lambda_{\mu'_-}^{\mu''} + (-)^{\eta} \delta_{\mu'_-}^{\mu''} \Lambda_{\mu}^{\mu'_-} \} \end{aligned} \quad (222)$$

having put  $\eta \equiv n_1 + n_1' + n'_1 + n'_1$ . From the results of Appendix one deduces that, indeed, the  $\frac{1}{2} r(r-1)$  independent operators  $\Lambda_{\mu}^{\mu'}$  are the generators of  $r$ -dimensional rotations namely, of the group  $R_r$ .

The Casimir operator of  $R_r$  is defined similarly to (203) for  $R_{2k+1}$  as

$$R_r: \quad \Phi = \sum_{\mu, \mu'} \Lambda_{\mu}^{\mu'} \Lambda_{\mu'}^{\mu} \quad (\mu, \mu' = 1, 2, \dots, r) \quad (223)$$

where

$$[\Phi, \Lambda_{\mu}^{\mu'}] = 0$$

and keeping in mind that  $\mu$  stands for  $(\eta, \eta_i, \eta_o)$ . Expanding this in a manner exactly analogous with the steps leading to (213) one arrives at

$$R_r: \quad \Phi = \frac{1}{2} \Gamma - P - \frac{1}{2} N \quad (224)$$

where

$$\Gamma = \sum_{\mu, \mu'} b_{\mu}^{\mu'} b_{\mu'}^{\mu}, \quad N = \sum_{\mu} b_{\mu}^{\mu}$$

refer, respectively, to the Casimir operator of  $R_r$  and the (invariant) particle operator;  $P$  is now the mixed-orbital pairing operator (219) which, being the linear combination of invariants.

$$P = \frac{1}{2} \Gamma - \Phi - \frac{1}{2} N, \quad (225)$$

is itself an invariant in the chain  $\mathcal{U}_r \supset R_r$ .

The eigenvalue  $\gamma$  of the operator  $\Gamma$  in a  $\mathcal{U}_r$  irreducible basis is, as in (205),

$$\gamma = \sum_{\mu=1}^r [h_{\mu}(h_{\mu} - 2\mu + r + 1)] \quad r = \frac{1}{2}(v+1)(v+2) \quad (226)$$

$V$  now referring to the  $V$ -shell orbital degeneracy. The eigenvalue  $\varphi$  of  $\Phi$  in an  $R_r$  irreducible basis will be (212), i.e.,

$$\varphi = \frac{1}{2} \sum_{\mu=1}^K \lambda_{\mu} (\lambda_{\mu} + r - 2\mu) \quad (227)$$

$$K = \frac{1}{2}(r-1) \text{ for } r \text{ odd} ; \quad K = \frac{1}{2}r \text{ for } r \text{ even.} \quad (228)$$

The various shell orbital-degeneracies  $r$  are given in Table .

$V$	SHELL	$r$
0	1s	1
1	1p	3
2	2s-1d	6
3	2p-1f	10
4	3s-2d-1g	15
5	3p-2f-1h	21
6	4s-3d-2g-1i	28

TABLE IV. 3b. 1

For any shell, the mixed-orbital pairing force has eigenvalue  $(\frac{1}{2} \gamma - \varphi - \frac{1}{2} N)$  ; hence

$$E_{(\lambda_1, \lambda_2, \dots, \lambda_K)}^{[f]} = \frac{1}{2} \left[ \sum_{\mu=1}^K h_{\mu} (h_{\mu} - 2\mu + r) - \sum_{\mu=1}^K \lambda_{\mu} (\lambda_{\mu} - 2\mu + r) \right] \quad (229a)$$

$$[f] \equiv [h_1, h_2, \dots, h_r] \quad (\text{orbital permutational symmetry})$$

taking care to follow the correct  $K$  values involved in (226). Finally, as each  $R_r$  irreducible representation  $(\lambda_1, \lambda_2, \dots, \lambda_K)$  will contain one or more  $R_3$  irreducible representations, namely  $L$ -values, one can label the spectrum (229) by total orbital angular momentum values which however will be degenerate. The basis is irreducible under transformations of the chain

$$U_r \supset R_r \supset R_3 \supset R_2 \quad (229b)$$

and is denoted by the set of kets

$$| [f] \alpha' (\lambda_1, \lambda_2, \dots, \lambda_K) \omega' L M_L \rangle \quad (229c)$$

where  $[f] \equiv [h_1, h_2, \dots, h_r]$  and  $(\alpha', \omega')$  are distinguishing labels as in (218).

The effect of an attractive pairing force between two particles in a single orbital  $\ell$  will be to lower the level  $\Lambda=0$  from the degenerate group of levels  $\Lambda=0, 1, 2, \dots, 2\ell$ , thus producing a "gap", as can be seen from the Racah matrix element on page 98 which we have used in our second-quantization formulation of the operator  $\mathcal{P}$ . But more importantly, its resemblance with a short-range central interaction as well as its group invariance properties warrants its use as a model interaction. The latter



characteristics we have just seen. Its short-range character is well displayed in some calculations of wave function overlaps, energy levels and quadrupole moments carried out parallelly between a gaussian central interaction and a "model" central interaction consisting of a variable mixture of  $P$  and  $Q^2$  forces. The cases covered by Mello<sup>64)</sup> were for 2,3 and 4 particles in the  $2s-1d$  nuclear shell using  $U_6$  partitions  $[f] = [2], [11], [3], [21], [111]$  and  $[4]$ . Energy level and quadrupole moment diagrams for both gaussian and model interactions show marked resemblances at intermediate ranges. In particular, overlaps calculated between wave functions of pure pairing against those of zero-range gaussian (which with proper normalization reduces to a  $\delta$ -force) are extremely good for the lowlying levels arising from the partitions  $[f]$  mentioned above. Good overlaps are also found in the comparison of pure  $Q^2$  and long range gaussian.

One undesirable though not very serious feature of the pairing force is its inability to break degeneracies as well as an ordinary  $\delta$ -force. One can see this, eg., in the comparison for  $N=2$  given by Biedenharn.<sup>65)</sup>

#### 4a. A Word About Exchange at Short-Ranges.

At short range, ideally a  $\delta$ -force, the Majorana exchange operator (178d) becomes the Wigner operator ( $\equiv 1$ ) and consequently the Bartlett and Heisenberg operators (178b,c) coincide.

Treatment of the Bartlett operator has proven difficult for

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short ranges, we shall thus make the not unreasonable assumption that at short ranges the Wigner part predominates over the Bartlett such as to justify neglecting the latter. Consequently, the effect of exchange at this extreme reduces to a constant factor to be ignored, as only energy differences arising from the pairing force matter.

### 5. Model Central Interaction Composed of $P$ and $Q^2$

Combining the interaction operators discussed in the previous sections one could therefore adopt the model central interaction (apart from an overall intensity factor  $V_0$ )

$$\chi P + (1-\chi) Q^2 \quad (0 \leq \chi \leq 1) \quad (230)$$

with the variable parameter  $\chi$  determining the percentage of  $P$  involved in the mixture.

To re-stress the group theoretic simplicity of the model interaction constituents  $P$  and  $Q^2$  we recall from (168) and (225) that, apart from the invariant number operator  $N$ ,

$$\begin{aligned} Q^2 &\longrightarrow \text{Casimir operators of } U_3 \text{ and } R_3 \\ P &\longrightarrow \text{Casimir operators of } U_r \text{ and } R_r. \end{aligned}$$

Consequently,

$$\begin{aligned} Q^2 &\text{ is diagonal in the scheme } U_3 \supset R_3 \\ P &\text{ is diagonal in the scheme } U_r \supset R_r. \end{aligned}$$

The first scheme can be extended to  $U_r \supset U_3 \supset R_3 \supset R_2$  to include permutational and axial symmetries; the second can be enlarged to  $U_r \supset R_r \supset R_3 \supset R_2$  to include spherical and axial symmetries. (These "enlargements" of the group chains will allow us to reduce

the number of states to be dealt with in the calculation of matrix elements.) If one decides upon one of the two schemes, the operator corresponding to it will be diagonal with eigenvalues for which simple algebraic formulas hold, i.e., (176) or (229). The remaining operator is not diagonal of course, and its matrix elements in the chosen basis must be evaluated. For reasons already stated in the Introduction we will here choose the  $U_3$ -scheme.

All interactions dealt with thus far are spin-independent. In the following section we introduce spin-orbit coupling and shall consider the evaluation of its matrix elements in the  $U_3$  scheme.

#### 6. The Spin-Orbit Force.

We are interested in the matrix elements of the single-body spin-orbit operator for N-particles

$$W_{s.o.} = \sum_{i=1}^N \vec{\ell}_i \cdot \vec{A}_i \quad (231)$$

calculated in the  $U_3$  scheme. In second-quantized language this operator becomes, by (24),

$$W_{s.o.} \leftrightarrow \mathcal{W}_{s.o.} = \sum_{\rho\rho'} \langle \rho | \vec{\ell}_i \cdot \vec{A}_i | \rho' \rangle C_{\rho}^{\rho'} \quad (232)$$

recalling that  $\rho = (\mu\sigma\tau)$  refer to orbital-spin-isopin single-particle quantum numbers and  $C_{\rho}^{\rho'}$  are the  $U_{4r}$  group generators.

The transformation properties of  $\mathcal{W}_{s.o.}$  under  $U_3$  will be specified and fully exploited. Accordingly, instead of using a basis set irreducible under transformations according to the chain (99) with  $U_3$  inserted between  $U_r$  and  $R_3$  as in (145), we shall

use the segment

$$U_r \supset U_3 \supset U_2 \supset U_1 \quad (\text{canonical}) \quad (233)$$

replacing the segment

$$U_r \supset U_3 \supset R_3 \supset R_2 \quad (\text{physical}) \quad (234)$$

in chain (99). The irreducible representations of  $U_3 \supset U_2 \supset U_1$  are characterized by  $(h_{13}, h_{23}, h_{33}, h_{12}, h_{22}, h_{11})$ . On the other hand, the weight  $(w_1, w_2, w_3)$  in  $U_3$  is given by

$$C_q^q \left| \begin{array}{ccc} h_{13} & h_{23} & h_{33} \\ & h_{12} & h_{22} \\ & & h_{11} \end{array} \right\rangle = w_q \left| \begin{array}{ccc} h_{13} & h_{23} & h_{33} \\ & h_{12} & h_{22} \\ & & h_{11} \end{array} \right\rangle \quad (235)$$

$$(q = 1, \bar{1}, 0 \rightarrow 1, 2, 3)$$

in accordance with definition (67) and numbering convention in  $q$  of Section 6. By (94a) one has explicitly the eigenvalues of  $C_1^1$ ,  $C_2^2$  and  $C_3^3$  respectively as

$$\begin{aligned} w_1 &= h_{11} \\ w_2 &= (h_{12} + h_{22}) - h_{11} \\ w_3 &= (h_{13} + h_{23} + h_{33}) - (h_{12} + h_{22}). \end{aligned} \quad (236)$$

Now, from the many-particle angular momentum operators  $L_q$  ( $q = 1, \bar{1}, 0$ ) expressed in terms of  $U_3$  generators  $C_q^{q'}$  ----- relations (142a)

----- we see that  $\mathcal{L}_2 = \mathcal{L}_0 = C_1' - C_1^{\bar{1}}$  will therefore have an eigenvalue

$$M_L = w_1 - w_2 = 2h_{11} - (h_{12} + h_{22})$$

so that classification by  $U_2 \supset U_1$  with  $(h_{12}, h_{22}, h_{11})$  is equivalent to classification by  $U_2 \supset R_2$  with  $(h_{12}, h_{22}, M_L)$ ,  $R_2$  being the group of rotations about the  $z$ -axis. For convenience, let us call

$$(h_{13}, h_{23}, h_{33}) \equiv (h_1, h_2, h_3)$$

$$(h_{12}, h_{22}) \equiv (q_1, q_2) \quad (237)$$

$$(h_{11}) \equiv (r_1) = \frac{1}{2}(M_L + q_1 + q_2)$$

so that our general state would be given by the set of kets

$$| [f] \alpha (h_1 h_2 h_3) q_1 q_2 M_L; \beta S M_S T M_T \rangle \quad (238)$$

where  $\{v\}$  has been suppressed as  $\{v\} = [\tilde{f}]$  makes it redundant.

Writing the operator (232) in more detail

$$\begin{aligned} W_{so}^p &= \sum_{\substack{\mu\sigma\tau \\ \mu'\sigma'\tau'}} \langle \mu\sigma\tau | \vec{L} \cdot \vec{S} | \mu'\sigma'\tau' \rangle C_{\mu\sigma\tau}^{\mu'\sigma'\tau'} \\ &= \sum_{q''} (-)^{q''} \sum_{\mu\mu'} \sum_{\substack{\sigma\sigma' \\ \tau}} \langle \mu | L_{-q''} | \mu' \rangle \langle \sigma | S_{q''} | \sigma' \rangle C_{\mu\sigma\tau}^{\mu'\sigma'\tau'} \end{aligned}$$

where  $q''$  designate the spherical component indices  $(1, \bar{1}, 0)$  and the unnecessary particle index is suppressed. Introducing expansion (118) for  $l-q''$  one gets

$$\begin{aligned} W_{S.O.} &= \sum_{q'q''} (-)^{q+q''} E_{-q''-q q'} \sum_{\sigma\sigma'} \langle \sigma | \Lambda_{q''} | \sigma' \rangle \sum_{\mu\mu'} \langle \mu | C_q^{q'} | \mu' \rangle C_{\mu\sigma\tau}^{\mu'\sigma'\tau} \\ &= \sum_{q'q''} (-)^{q+q''} E_{-q''-q q'} \sum_{\sigma\sigma'} \langle \sigma | \Lambda_{q''} | \sigma' \rangle C_{q\sigma\tau}^{q'\sigma'\tau} \quad (239) \end{aligned}$$

where the set of operators  $C_{q\sigma\tau}^{q'\sigma'\tau}$  can be interpreted as the generators of a subgroup  $U_3 \times U_2(\text{spin}) \times U_2(\text{isospin})$  of  $U_{4r}$  and  $U_3$  being the subgroup of  $U_r$  discussed in Chapter III,6. The set  $\{ C_{q\sigma\tau}^{q'\sigma'\tau} \}$  from the above equations, is hereby expressed as linear combinations of the  $U_{4r}$  generators  $C_{\mu\sigma\tau}^{\mu'\sigma'\tau}$ , namely

$$C_{q\sigma\tau}^{q'\sigma'\tau} = \sum_{\mu\mu'} \langle \mu | C_q^{q'} | \mu' \rangle C_{\mu\sigma\tau}^{\mu'\sigma'\tau}. \quad (240)$$

If one now defines a tensor (traceless with respect to indices  $q, q' = 1, \bar{1}, 0$ )

$$T((2)_q^{q'}; 1_{q''}) \equiv \sum_{\sigma\sigma'} \langle \sigma | \Lambda_{q''} | \sigma' \rangle \left[ C_{q\sigma\tau}^{q'\sigma'\tau} - \frac{1}{3} \left( \sum_{\bar{q}} C_{\bar{q}\sigma\tau}^{\bar{q}'\sigma'\tau} \right) \delta_q^{q'} \right], \quad (241)$$

then from (239) and the fact that  $\sum_{q'q''} (-)^{q+q''} E_{-q''-q q'} \delta_q^{q'}$  vanishes we have simply that

$$W_{S.O.} = \sum_{q'q''} (-)^{q+q''} E_{-q''-q q'} T((2)_q^{q'}; 1_{q''}). \quad (242)$$

Labelling  $(21)_{\frac{1}{2}}^{q'}; 1_{\frac{1}{2}}^{q'}$  on tensor  $\Pi$  refers to its transformation properties under  $SU_3$  and  $U_2(\text{spin})$ , respectively, and will now be explained. This can be seen from the commutators between the  $U_3$  generators  $C_q^{q'}$  and the operators (240). From (131), (240), (37) (32) and (120) they are found to be

$$\begin{aligned} [C_{\bar{q}}^{\bar{q}'}, C_{q\sigma\tau}^{q'\sigma'\tau'}] &= \left[ \sum_{\bar{\mu}\bar{\mu}'} \langle \bar{\mu} | C_{\bar{q}}^{\bar{q}'} | \bar{\mu}' \rangle C_{\bar{\mu}}^{\bar{\mu}'}, \sum_{\mu\mu'} \langle \mu | C_q^{q'} | \mu' \rangle C_{\mu s}^{\mu' s'} \right] \\ &= \sum_{\substack{\mu\mu' \\ \bar{\mu}\bar{\mu}'}} \langle \bar{\mu} | C_{\bar{q}}^{\bar{q}'} | \bar{\mu}' \rangle \langle \mu | C_q^{q'} | \mu' \rangle \sum_{\bar{s}s'} \delta_{\bar{s}}^{\bar{s}'} \{ C_{\bar{\mu}\bar{s}}^{\mu' s'} \delta_{\mu}^{\bar{\mu}'} \delta_{s'}^{\bar{s}'} - C_{\mu s}^{\bar{\mu}' \bar{s}'} \delta_{\bar{\mu}}^{\mu'} \delta_s^{\bar{s}'} \} \end{aligned}$$

$$[C_{\bar{q}}^{\bar{q}'}, C_{q\sigma\tau}^{q'\sigma'\tau'}] = C_{\bar{q}\sigma\tau}^{q'\sigma'\tau'} \delta_{\bar{q}}^{\bar{q}'} - C_{q\sigma\tau}^{\bar{q}'\sigma'\tau'} \delta_{\bar{q}}^{\bar{q}'} \quad (243)$$

since  $S=(\sigma\tau)$ . Since (243) is similar to relations (135) comprising the Lie algebra of  $U_3$ , operators  $C_{q\sigma\tau}^{q'\sigma'\tau'}$  transform under  $U_3$  like the generators  $C_q^{q'}$ . But these in turn transform exactly like the single-particle generators  $C_q^{q'} \equiv a_q^+ a^q$  or, as

$$\square \otimes \bar{\Pi} = \bar{\Pi} + \bar{\Pi} = (2\bar{0}) + \quad (111),$$

where  $\square$  represents a quanta from  $a_q^+$  and  $\bar{\Pi}$  a hole from  $a^q$ , the " $\otimes$ " representing the outer product of two representations which is



calculated by Little wood's rules for unitary groups (see Hammermesh, ref. 34, p. 249). Hence  $\left[ C_{q\sigma\tau}^{q'\sigma'\tau'} - \frac{1}{3} \left( \sum_{\bar{q}} C_{\bar{q}\sigma\tau}^{q'\sigma'\tau'} \right) \delta_{\bar{q}}^{q'} \right]$  transforms under  $U_3$  like  $(h_1, h_2, h_3) = (210)$  since the trace  $\left( \sum_{\bar{q}} C_{\bar{q}\sigma\tau}^{q'\sigma'\tau'} \right)$  transforms like  $(h_1, h_2, h_3) = (000)$ . Therefore,  $\Pi((2)_q^{q'}; 1_{q''})$  transforms irreducibly under  $U_3$  like  $(k_1, k_2) = (21)$  with indices  $q, q'$  which determine the row of that representation. The coefficients in (241) can be written, by the Wigner-Eckhart theorem, as

$$\begin{aligned} \langle \sigma | \Delta_{q''}^{(1)} | \sigma' \rangle &\equiv \langle \frac{1}{2} \sigma | \Delta_{q''}^{(1)} | \frac{1}{2} \sigma' \rangle \\ &= \langle \frac{1}{2} 1 \sigma' q'' | \frac{1}{2} \sigma \rangle \langle \frac{1}{2} \| \Delta^{(1)} \| \frac{1}{2} \rangle \quad (244) \end{aligned}$$

the superscript 1 indicating  $\Delta_{q''}^{(1)}$  to be a tensor of rank 1 (a vector) and of projection  $q'' = 1, \bar{1}, 0$  so that, under  $U_2(\text{spin})$ , (241) transforms irreducibly like a vector with projection index  $q''$ .

Before proceeding to the evaluation of  $W_{s.o.}$  in matrix from between states (238), we require an important result to be sketched below.

---

Theorem. Recall first of all that any vector operator  $X_q$  ( $q=1, \bar{1}, 0$ ) obeys with  $R_3$  generators  $\mathcal{L}_q$  ( $q=1, \bar{1}, 0$ ) commutation relations  $[\mathcal{L}_0, X_{\pm 1}] = \pm X_{\pm 1}$  and  $[\mathcal{L}_1, X_{\bar{1}}] = -X_0$  similar to relations (142b) between  $\mathcal{L}_q$  themselves. Consequently, the  $X_q$  transform irreducibly under  $R_3$  like the  $\mathcal{L}_q$ . The Wigner-Eckhart theorem thus tells us that  $\langle LM' | X_q | LM \rangle = A_L \langle LM' | \mathcal{L}_q | LM \rangle$  where the proportionality constant  $A_L$  is

independent of the row of  $R_3$ , namely  $M$  and  $M'$ . This constant is furthermore easy to evaluate as for  $M=M'=L$  and  $q=0$  we have  $A_L = \frac{1}{L} \langle L L | X_0 | L L \rangle$ . Vectorially, we have added  $\vec{L} + \vec{1}$  to give  $\vec{L}$  which appears only once.

For the case of  $U_3$ , consider the operator set  $\{Z_q^{q'}\}$  which obeys the commutation relations

$$[C_q^{q'}, Z_{q''}^{q'''}] = Z_q^{q'''} \delta_{q''}^{q'} - Z_{q''}^{q'} \delta_q^{q'''} \quad (245)$$

so that  $Z_q^{q'}$  transforms like  $C_q^{q'}$ , specifically like  $(h_1, h_2, h_3) = (2, 1, 0)$  or  $(k_1, k_2) = (2, 1)$  under  $SU_3$ . It is easily seen that since

$$\text{Tr } Z = \sum_q Z_q^q \quad \text{and} \quad [C_q^{q'}, \text{Tr } Z] = 0$$

then the traceless operators

$$\bar{Z}_q^{q'} \equiv Z_q^{q'} - \frac{1}{3} (\text{Tr } Z) \delta_q^{q'} \quad (246)$$

satisfy the same commutation relations (245) and transform like (21) of  $SU_3$ . If one is interested in the matrix elements of  $\bar{Z}_q^{q'}$ , in the chain  $U_3 \supset U_2 \supset U_1$ , between the same  $U_3$  representation, that is,

$$\left\langle \begin{array}{ccc} h_{13} & h_{23} & h_{33} \\ h'_{12} & h'_{22} & \\ h''_{11} & & \end{array} \right| \bar{Z}_q^{q'} \left| \begin{array}{ccc} h_{13} & h_{23} & h_{33} \\ h_{12} & h_{22} & \\ h_{11} & & \end{array} \right\rangle, \quad (247)$$

one must consider the fact that, by Littlewood's rules, the outer product

$$(h_{13} h_{23} h_{33}) \otimes (210) = \dots + (h_{13} h_{23} h_{33})^2 + \dots$$

involved in (247) gives, among other representations, the

$(h_{13} h_{23} h_{33})$  twice except when  $h_{33} = h_{23}$  or  $h_{23} = h_{13}$ . Thus, (247) is expandable as the matrix elements of two independent operator sets which must transform under  $U_3$  like (210). We saw that the traceless generators

$$\bar{C}_q^{q'} \equiv C_q^{q'} - \frac{1}{3} (\text{Tr } C) \delta_q^{q'} \quad (248)$$

form one such set. The traceless operator set independent from the above, namely,

$$\bar{G}_q^{q'} \equiv \sum_{q''} C_q^{q''} C_{q'}^{q'} - \frac{1}{3} \left( \sum_{q'' q'''} C_q^{q''} C_{q'''}^{q''} \right) \delta_q^{q'} \quad (249)$$

also transforms like (21) under  $SU_3$  as it is easily shown that

$$\begin{aligned} [\bar{C}_q^{q'}, \bar{G}_{\bar{q}}^{\bar{q}'}] &= [C_q^{q'}, \sum_{q''} C_{\bar{q}}^{q''} C_{q''}^{\bar{q}'}] \\ &= \bar{G}_q^{\bar{q}'} \delta_{\bar{q}}^{q'} - \bar{G}_{\bar{q}}^{q'} \delta_q^{\bar{q}'} \end{aligned} \quad (250)$$

Hence, one can write (247) as

$$\left\langle \begin{array}{c} h_{13} h_{23} h_{33} \\ h'_{12} h'_{22} \\ h''_{11} \end{array} \right| \bar{Z}_q^{q'} \left| \begin{array}{c} h_{13} h_{23} h_{33} \\ h_{12} h_{22} \\ h_{11} \end{array} \right\rangle =$$

(251)

$$A \begin{array}{c} \left\langle \begin{array}{ccc} h_{13} & h_{23} & h_{33} \\ h'_{12} & h'_{22} & \\ h'_{11} & & \end{array} \right| \bar{G}_q^{q'} \left| \begin{array}{ccc} h_{13} & h_{23} & h_{33} \\ h_{12} & h_{22} & \\ h_{11} & & \end{array} \right\rangle + B \begin{array}{c} \left\langle \begin{array}{ccc} h_{13} & h_{23} & h_{33} \\ h'_{12} & h'_{22} & \\ h'_{11} & & \end{array} \right| \bar{G}_q^{q'} \left| \begin{array}{ccc} h_{13} & h_{23} & h_{33} \\ h_{12} & h_{22} & \\ h_{11} & & \end{array} \right\rangle \end{array}$$

the A and B coefficients being independent of the row of  $U_3$ . These are evaluated by taking  $q = q'$  and maximum weight in both bras and kets, i.e., for

$$h_{12} = h_{11} = h'_{12} = h'_{11} = h_{13}$$

$$h_{22} = h'_{22} = h_{23},$$

from which considerations three inhomogeneous linear equations result, one of which is dependent as

$$\text{Tr}(\bar{Z}) = \text{Tr}(\bar{C}) = \text{Tr}(\bar{G}) \equiv 0.$$

It is simpler, therefore, to evaluate A and B via the 2 independent equations involving matrix elements of

$$\bar{Z}_1' - \bar{Z}_2^2 = Z_1' - Z_2^2 \quad \text{and} \quad \bar{Z}_2^2 - \bar{Z}_3^3 = Z_2^2 - Z_3^3.$$

These results are generalizable to the  $U_r$  group, should they be required for a given problem.

---

Returning to our specific problem, the matrix elements of (241) between states (238) with the same  $[f] \alpha(h_1 h_2 h_3)$  in bra and ket will be, in accordance with result (251),

$$\begin{aligned} & \langle [f] \alpha(h_1 h_2 h_3) q_1' q_2' M_L'; \beta' S' M_S' T | \mathbb{I}((21)_1^q; 1_{q''}) | [f] \alpha(h_1 h_2 h_3) q_1 q_2 M_L; \beta S M_S T \rangle \\ &= \left\{ A_{S'S} \langle h_1 h_2 h_3, q_1' q_2', M_L' | \bar{C}_q^{q'} | h_1 h_2 h_3, q_1 q_2, M_L \rangle \right. \\ & \quad \left. + B_{S'S} \langle h_1 h_2 h_3, q_1' q_2', M_L' | \bar{G}_q^{q'} | h_1 h_2 h_3, q_1 q_2, M_L \rangle \right\} \langle S 1 M_S q'' | S' M_S' \rangle \end{aligned} \quad (252)$$

where the Wigner - Eckhart theorem has been used in the spin part to give on the right hand side

$$\langle S 1 M_S q'' | S' M_S' \rangle \langle \frac{1}{2} \| J^{(1)} \| \frac{1}{2} \rangle = \langle S 1 M_S q'' | S' M_S' \rangle \sqrt{\frac{3}{4}},$$

the factor  $\langle \frac{1}{2} \| J^{(1)} \| \frac{1}{2} \rangle = \sqrt{J(J+1)} = \sqrt{3/4}$  being incorporated into the coefficients  $A_{S'S}$  and  $B_{S'S}$  which depend neither on  $q_1 q_2 M_L$  (the row of  $U_3$ ) nor on  $M_S$  (the row of  $U_2$ ), but do depend on  $[f] \alpha(h_1 h_2 h_3)$  of  $\mathcal{U}_r > U_3$ .

Coefficients  $A_{S'S}$  and  $B_{S'S}$  can now be evaluated by constructing the two linearly independent inhomogeneous equations corresponding to the matrix elements of the two differences

$$\mathbb{I}((21)_1^q; 1_{q''}) - \mathbb{I}((21)_2^q; 1_{q''}) \quad \text{and} \quad \mathbb{I}((21)_2^q; 1_{q''}) - \mathbb{I}((21)_3^q; 1_{q''}) \quad (253 a, b)$$

(as before:  $q, q' = 1, \bar{1}, 0 \rightarrow 1, 2, 3$ ) between states of the same  $U_3$  representation  $(h_1, h_2, h_3)$  on both bra and ket, of max. wt. in  $U_3 \supset U_2 \supset U_1$  and max. proj. of spin, namely for:

$$(h'_1, h'_2, h'_3) = (h_1, h_2, h_3)$$

$$h_1 = q_1 = q'_1 = r_1 = r'_1$$

$$h_2 = q_2 = q'_2$$

(254)

$$M'_S = S'$$

$$M_S = S$$

$$\therefore q'' = M'_S - M_S = S' - S.$$

The calculation is simple and will be given in Chapter VIII for the case  $[f](h_1, h_2, h_3) = [31](710)$  corresponding to the lowest levels of  $F^{20}$  in the  $U_3$  scheme.

From (242) and (252) the spin-orbit operator  $\mathcal{W}_{S.O.}^P$  for N particles will thus have matrix elements between states (238) of the same  $[f] \propto (h_1, h_2, h_3)$  given by

$$\begin{aligned} & \langle [f] \propto (h_1, h_2, h_3) q'_1 q'_2 M'_L; \beta' S' M'_S T | \mathcal{W}_{S.O.}^P | [f] \propto (h_1, h_2, h_3) q_1 q_2 M_L; \beta S M_S T \rangle \\ &= \sum_{q''} (-1)^{q''} \left\{ A_{S'S} \langle h_1, h_2, h_3, q'_1 q'_2, M'_L | \sum_{q, q'} (-1)^q \epsilon_{-q q q'} \bar{C}_q^{q'} | h_1, h_2, h_3, q_1 q_2, M_L \rangle \right. \\ & \quad \left. + B_{S'S} \langle h_1, h_2, h_3, q'_1 q'_2, M'_L | \sum_{q, q'} (-1)^q \epsilon_{q'' - q q'} \bar{C}_q^{q'} | h_1, h_2, h_3, q_1 q_2, M_L \rangle \right\} \\ & \quad \times \langle S' M_S q'' | S' M'_S \rangle \end{aligned} \quad (255)$$

But, from (141) we have that the operator

$$\sum_{q q'} (-)^q \epsilon_{-q'' - q q'} \bar{C}_q^{q'} = L_{-q''}^{(1)},$$

the trace term of  $\bar{C}_q^{q'} \equiv C_q^{q'} - \text{Tr}(C) \delta_q^{q'}$  contributing with zero as before, where of course  $L_{-q''}^{(1)}$  transforms like a vector (superscript 1). Calling the second operator on the right - hand side of (255)

$$K_{-q''} \equiv \sum_{q q'} (-)^q \epsilon_{-q'' - q q'} \bar{C}_q^{q'} \quad (256)$$

it is easily seen that it too transforms like a vector since the commutator

$$[L_q, K_{-q''}] = \sum_{q'} (-)^{q'} \epsilon_{q - q'' q'} K_{-q''}$$

is precisely the condition that  $K_{-q''}$  be an irreducible Racah tensor under  $R_3$  (compare with Rose, Angular Momentum p.84, ref. (35)). One is thus left with the problem of evaluating matrix elements

$$\langle h_1 h_2 h_3, q'_1 q'_2, M'_L | L_{-q''}^{(1)} | h_1 h_2 h_3, q_1 q_2, M_L \rangle$$

$$\langle h_1 h_2 h_3, q'_1 q'_2, M'_L | K_{-q''}^{(1)} | h_1 h_2 h_3, q_1 q_2, M_L \rangle$$

in the basis irreducible under  $U_3 \supset U_2 \supset U_1$ . The similarity transformation passing from this basis to the basis irreducible under  $U_3 \supset R_3 \supset R_2$  with respective quantum labels (145 b)

and given by the kets

$$|(h_1 h_2 h_3) \omega L M_L\rangle \equiv |(k, k_2) \omega L M_L\rangle \begin{cases} k_1 \equiv h_1 - h_3 \\ k_2 \equiv h_2 - h_3 \end{cases}$$

(as in eqns. 176 and 177) can be applied to both sides of (255). This transformation will be discussed in detail later. For the moment, if one imagined it applied to (255) the result would evidently be

$$\begin{aligned} & \langle [f] \alpha(k, k_2) \omega' L' M_L'; \beta' S' M_S' T | \mathcal{W}_{S,0}^{(1)} | [f] \alpha(k, k_2) \omega L M_L; \beta S M_S T \rangle \\ &= \sum_{q''} (-1)^{q''} \left\{ A_{S'S} \langle (k, k_2) \omega' L' M_L' | \mathcal{L}_{-q''}^{(1)} | (k, k_2) \omega L M_L \rangle \right. \\ & \quad \left. + B_{S'S} \langle (k, k_2) \omega' L' M_L' | \mathcal{K}_{-q''}^{(1)} | (k, k_2) \omega L M_L \rangle \right\} \langle S 1 M_S q'' | S' M_S' \rangle \end{aligned} \quad (257)$$

$\omega, \omega'$  having the same significance as in chain (145). The matrix elements of  $\mathcal{L}_{-q''}^{(1)}$  and  $\mathcal{K}_{-q''}^{(1)}$  can now be reduced by the Wigner-Eckhart theorem in  $R_3$  thusly:

$$\begin{aligned} & \langle (k, k_2) \omega' L' M_L' | \mathcal{L}_{-q''}^{(1)} | (k, k_2) \omega L M_L \rangle = \\ & \langle L 1 M_L - q'' | L' M_L' \rangle \langle (k, k_2) \omega' L' || \mathcal{L}^{(1)} || (k, k_2) \omega L \rangle \\ & \langle (k, k_2) \omega' L' M_L' | \mathcal{K}_{-q''}^{(1)} | (k, k_2) \omega L M_L \rangle = \\ & \langle L 1 M_L - q'' | L' M_L' \rangle \langle (k, k_2) \omega' L' || \mathcal{K}^{(1)} || (k, k_2) \omega L \rangle \end{aligned}$$

with the same Clebsch-Gordon coefficient as proportionality constant.



Furthermore, the reduced matrix elements are:

$$\begin{aligned}
 \langle (k_1 k_2) \omega' L' \| \mathcal{L}^{(q)} \| (k_1 k_2) \omega L \rangle &= \frac{\langle (k_1 k_2) \omega' L' L' | \mathcal{L}_{L'-L}^{(q)} | (k_1 k_2) \omega L L \rangle}{\langle L 1 L L' - L | L' L' \rangle} \\
 &= \frac{\langle (k_1 k_2) \omega' L' L | \mathcal{L}_0^{(q)} | (k_1 k_2) \omega L L \rangle}{\langle L 1 L 0 | L' L \rangle} \\
 &= \sqrt{L(L+1)} \delta_{\omega' \omega} \delta_{L' L}
 \end{aligned}$$

and, for  $\mathcal{K}^{(q)}$ , arbitrarily choosing  $M_L' = M_L = 1$  one has

$$\begin{aligned}
 \langle (k_1 k_2) \omega' L' \| \mathcal{K}^{(q)} \| (k_1 k_2) \omega L \rangle &= \frac{\langle (k_1 k_2) L' 1 | \mathcal{K}_0^{(q)} | (k_1 k_2) L 1 \rangle}{\langle L 1 1 0 | L' 1 \rangle} \\
 &= \frac{1}{\langle L 1 1 0 | L' 1 \rangle} \sum_{\substack{q_1 q_2 \\ q'_1 q'_2}} \langle \omega' L' (k_1 k_2 M_L = 1) q'_1 q'_2 \rangle \langle q'_1 q'_2 1 | \mathcal{K}_0^{(q)} | q_1 q_2 1 \rangle \\
 &\quad \times \langle q_1 q_2 (k_1 k_2 M_L = 1) \omega L \rangle
 \end{aligned} \tag{258}$$

where the bracket symbols<sup>66)</sup>  $\langle \omega' L' (k_1 k_2 M_L) q'_1 q'_2 \rangle$  stand for the elements of the similarity transformation matrix that passes a basis from  $U_3 \supset U_2 \supset U_1$  to  $U_3 \supset R_3 \supset R_2$ , i.e., from the "canonical" to the "physical" chain. One therefore needs first to evaluate (for  $M_L = M_L' = 1$ ) the matrix with elements

$$\langle (k, k_2) q_1' q_2' M_L' | \mathcal{K}_0^{(1)} | (k, k_2) q_1 q_2 M_L \rangle \quad (259)$$

where by (256)  $\mathcal{K}_0^{(1)} = \bar{G}_1' - \bar{G}_2^2 = G_1' - G_2^2$  and by definition (249)

$$\mathcal{K}_0^{(1)} = C_1' C_1' + C_1^2 C_2' + C_1^3 C_3' - C_2' C_1^2 - C_2^2 C_2^2 - C_2^3 C_3^2 \quad (260)$$

so that matrix elements (259) can be found if one has the matrix elements of  $U_3$  generators  $C_q^{q'}$  ( $q, q' = 1, \bar{1}, 0 \rightarrow 1, 2, 3$ ) in the chain  $U_3 \supset U_2 \supset U_1$  since, for instance, suppressing labels  $(k, k_2)$ ,

$$\langle (k, k_2) q_1' q_2' M_L' | C_1^2 C_2' | (k, k_2) q_1 q_2 M_L \rangle = \quad (261)$$

$$\sum_{\bar{q}_1, \bar{q}_2, \bar{M}_L} \langle q_1' q_2' M_L' | C_1^2 | \bar{q}_1, \bar{q}_2, \bar{M}_L \rangle \langle \bar{q}_1, \bar{q}_2, \bar{M}_L | C_2' | q_1 q_2 M_L \rangle$$

is a typical term. The elements  $\langle q_1' q_2' M_L' | C_q^{q'} | q_1 q_2 M_L \rangle$  (for  $q, q' = 1, \bar{1}, 0 \rightarrow 1, 2, 3$ ) are listed in Appendix H. Using these results, after a brief calculation (259) becomes: (see next page)

$$\langle (k, k_2) q_1' q_2' M_L' | \mathcal{L}_0^{(1)} | (k, k_2) q_1 q_2 M_L \rangle =$$

$$\sum_{q_1' q_2'} \delta_{q_1' q_2'} \sum_{M_L' M_L} M_L \left[ \frac{(q_1 + q_2 + 1)}{(q_1 - q_2)(q_1 - q_2 + 1)} + \frac{(k_1 - q_1 + 1)(q_1 - k_2)(q_1 + 1)}{(q_1 - q_2)(q_1 - q_2 + 1)} - \frac{q_2(k_2 - q_2 + 1)(k_1 - q_2 + 2)}{(q_1 - q_2 + 1)(q_1 - q_2 + 2)} \right]$$

$$- \sum_{q_1' q_2'} \delta_{q_1' q_2'} \sum_{M_L' M_L} \int \frac{(k_1 - q_1)(q_1 - k_2 + 1)(q_1 + 2) q_2 (k_2 - q_2 + 1)(k_1 - q_2 + 2)(q_1 - q_2 + M_L + 2)}{(q_1 - q_2 + 1)(q_1 - q_2 + 2)^2 (q_1 - q_2 + 3)} \quad (262)$$

$$- \sum_{q_1' q_2'} \delta_{q_1' q_2'} \sum_{M_L' M_L} \int \frac{(q_2 + 1)(k_2 - q_2)(k_1 - q_2 + 1)(k_1 - q_1 + 1)(q_1 - k_2)(q_1 + 1)(q_1 - q_2 + M_L)(q_1 - q_2 - M_L)}{(q_1 - q_2 - 1)(q_1 - q_2)^2 (q_1 - q_2 + 1)}$$

$$k \geq q_1 \geq k_2 \geq q_2 \geq 0, \quad q_1 \geq q_1' \geq q_2, \quad q_1' \equiv q_1 + q_2 + M_L/2$$

which despite its formidable aspect is quite simple to evaluate for particular values of  $(k, k_2)$ . The inequality conditions listed below are the equivalent of (92) for  $U_3 > U_2 > U_1$ .

Combining these results---except for the last one regarding the vector  $\mathcal{K}^{(n)}$  which will be illustrated in application---and coupling  $\vec{L} + \vec{S}$  to give  $\vec{J}$  in the bra and ket of  $\mathcal{W}_{s.o.}$  by Clebsch - Gordon coefficients, one obtains

$$\begin{aligned}
 & \langle [f] \alpha(k, k_2) \omega' L'; \beta' S', J M_J, T | \mathcal{W}_{s.o.} | [f] \alpha(k, k_2) \omega L, \beta S, J M_J, T \rangle \\
 &= \sum_{q'' M_L M_S} (-1)^{q''} \langle L S M_L M_S | J M_J \rangle \langle L' S' M_L' M_S' | J M_J \rangle \langle L' M_L - q'' | L' M_L' \rangle \\
 & \quad \times \langle S' M_S q'' | S' M_S' \rangle \left\{ A_{S'S} \sqrt{L(L+1)} \delta_{\omega' \omega} \delta_{L'L} \right. \\
 & \quad \left. + B_{S'S} \langle (k, k_2) \omega' L' || \mathcal{K} || (k, k_2) \omega L \rangle \right\} \\
 &= (-1)^{L'+S-J} \sqrt{(2L'+1)(2S'+1)} W(L'S'; LS; J1) \times \quad (263) \\
 & \quad \times \left\{ A_{S'S} \sqrt{L(L+1)} \delta_{\omega' \omega} \delta_{L'L} + B_{S'S} \langle (k, k_2) \omega' L' || \mathcal{K} || (k, k_2) \omega L \rangle \right\}
 \end{aligned}$$

after a slight rearrangement of  $\langle S' M_S q'' | S' M_S' \rangle$  with the aid of Clebsch - Gordon coefficient symmetry rules.  $W(L'S'; LS; J1)$  is the well-known Racah coefficient which can be defined as a sum over four Clebsch - Gordon coefficients (c.f. Rose, page 110). Matrix element formula (263) can be considered a generalization of Racah's formula<sup>60)</sup> for the same operator acting for N particles

all in the same orbital level 1.

In conclusion: evaluation of the spin - orbit interaction matrices in the  $SU_3$  scheme is reduced to:

1) evaluation of coefficients  $A_{S'S}$  and  $B_{S'S}$ .

2) evaluation of the quantities  $\langle (k, k_2) L' || \mathcal{K} || (k, k_2) L \rangle$  forming a matrix of rows and columns given by  $(L', L)$  with only the principal diagonal and the diagonals immediately above and below it differing from zero from the selection rule  $L' = L \pm 1$ ,  $L$  as  $K$  is a vector.

3) Use of tabulated Racah coefficients  $W(L'S', LS; J1)$ .

Points (1) and (2) will be seen in detail in application (Chap. VIII).

For particular  $SU_3$  representations  $(k, k_2)$  in which  $k_2 = 0$  or  $k_1 = k_2$ , the second term within the brackets of (263) vanishes thus leaving the problem of evaluating spin-orbit effects an extremely simple one (e.g.,  $F^N$  with  $N=2$  in the  $2s-1d$  shell,  $(k, k_2) = (40)$ , is such a case).

Finally, the case of different values of  $(k, k_2)$  between bra and ket of  $\mathcal{W}_{S.O.}$  --- which is outside our present scope --- is a problem involving actual use of  $SU_3$  Wigner coupling coefficients and is being studied.

# 7. The Complete Model Interaction.

The full hamiltonian in second-quantization form (143b) for N particles in an unfilled shell is now explicitly

$$H = H^{osc} + H_{int}$$

$$H^{osc} \rightarrow \sum_{i=1}^N \left\{ \frac{p_i^2}{2m} + \frac{1}{2} m \omega^2 r_i^2 \right\} \quad (264 \text{ a, b, c})$$

$$H_{int} = - (x P + y Q^2 + z W_{s.o.})$$

and since  $H^{osc}$  does not affect the relative spacing between levels it is thus neglected. Only  $H_{int}$  is pertinent. Factoring out  $(x+y+z)$  one has

$$H_{int} = -(x+y+z) \left[ \frac{x}{x+y+z} P + \frac{y}{x+y+z} Q^2 + \frac{z}{x+y+z} W_{s.o.} \right]$$

where the sum of the three coefficients is obviously unity. Letting

$$(x+y+z) = V_0, \quad \frac{x}{x+y+z} = \alpha, \quad \frac{y}{x+y+z} = \beta$$

then

$$\frac{z}{x+y+z} = 1 - \alpha - \beta$$

and since  $\alpha, \beta$  and  $(1-\alpha-\beta)$  must be positive or zero for  $x, y$  and  $z$  positive or zero one obtains

$$H_{int} = -V_0 [\alpha P + \beta Q^2 + (1-\alpha-\beta) W_{s.o.}]$$

$$\alpha + \beta \leq 1 \quad (265 \text{ a, b})$$

Coefficient parameters  $\alpha$  and  $\beta$  will be varied in compliance with (265 b) and parameter  $V_0$  determined by a least-mean-squares fit to

the empirical spectrum of the eigenvalues from the matrix

$$\| \langle [f] \alpha(k, k_2) \omega_L; \beta S'; J T | H_{int}(\mathcal{U}_0, \alpha, \beta) [f] \alpha(k, k_2) \omega_L; \beta S; J T \rangle \| \quad (266)$$

NOTE: distinguish parameters  $\alpha, \beta$  from quantum nos.  $\alpha, \beta$ .

having taken  $M_J = J$  and  $M_T = T$  because of invariance under rotations in the spin-orbital space and neglect of Coulomb effects.

Each predicted level is labelled by  $J, T$  and of course  $\pi$ , the parity, which is positive for excitations within the  $2s-1d$  shell. It is expected that for  $F^{20}$  the lowest levels are of  $T=1$ , the lowest  $T=2$  levels being above 6 Mev.

We know that the  $Q^2$  term in  $H_{int}$  of (264 c) is diagonal in the  $SU_3$  scheme states of (266). The matrix elements of  $P$  and  $W_{s.o.}$  must then be computed in this scheme, should it be chosen. For the spin-orbit matrix elements one only requires the maximum weight component of the  $U_3 \supset U_2 \supset U_1$  base. The whole base is needed to evaluate  $\|P\|$  but fortunately one only needs the diagonal elements for the case of  $F^{20}$ . These details will be discussed in application.

Alternatively, one could in principle choose the "pairing scheme" associated with the group  $R_7$  under which  $\|P\|$  diagonal but not  $\|Q^2\|$  and  $\|W_{s.o.}\|$ . Thirdly, a "f-f N - particle coupling scheme" which diagonalizes  $\|W_{s.o.}\|$  could be decided upon and under which the matrix elements of  $\|P\|$  and  $\|Q^2\|$  would have to be calculated.

All three coupling schemes will be discussed in the following chapter even though only the  $SU_3$  scheme will be adopted for our present calculations and these are relegated to Chapter IX.

V. POSSIBLE CLASSIFICATION SCHEMES FOR MANY - PARTICLE STATES IN AN OSCILLATOR SHELL, IN PARTICULAR THE  $2s-1d$  SHELL.

1. The  $Q^2$  (or  $SU_3$  group) Scheme.

The problem of finding the irreducible representations of  $U_3$  contained under a given one of  $U_r$  is similar to the chain calculation illustrated in Appendix A for the reduction  $U_r \supset R_3$  ( $r=6$ ). Further details, as well as an extensive table for the  $2s-1d$  shell, are found in Elliott.<sup>12)</sup> Finding the  $L$ -values contained in a given  $(k_1, k_2)$  representation of  $U_3$ , i.e., the reduction  $U_3 \supset R_3$ , are easily gotten by simple inequalities derived rigorously by Bargmann & Moshinsky<sup>67)</sup> and tabulated elsewhere<sup>68)</sup> by the same authors for some particular cases. The inequalities involved are simply

$$\underline{(k_1-L) \text{ even:}} \quad 0 \leq 2q \leq k_2 \quad k_2-L \leq 2q \leq k_1-L \quad (267)$$

$$\underline{(k_1-L) \text{ odd:}} \quad 0 \leq 2q \leq k_2-1 \quad k_2-L \leq 2q \leq k_1-L-1$$

where  $q$  is a non-negative integer.\* Using Elliott's and Bargmann - Moshinsky's tables and inequalities (267) we give

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\* The  $L$ -structure of a given  $(k_1, k_2)$  representation further breaks down into classes of levels, each class labelled by a number  $K \geq 0$  and which for quanta  $(k_1 \geq 2k_2)$  is simply  $K = k_2 - 2q$  while for quanta - holes  $(k_1 < 2k_2)$  it is  $K = k_1 - k_2 - 2q$ . Elliot<sup>12)</sup> identifies this  $K$  with the rotational - band quantum number  $K$  of the collective model (Figure 2).



Table V.1.1 covering  $N=1$  to 12 particles for the more symmetrical orbital partitions  $[f]$  and the lower (energywise)  $SU_3$  representations  $(k, k_2)$ . For  $N \leq 5$ , all  $SU_3$  representations present are given.

Some observations regarding these results are in order:

- 1) The maximum number of allowed nucleons in the  $2s-1d$  shell being  $4r = 24$ , only  $N \leq 12$  are included in Table since any number above that can be treated as  $N \leq 12$  "holes". Interaction matrix elements for  $N$  holes differ from those for  $N$  particles by simple phases.<sup>69)</sup>
- 2) Always  $L \leq k_1$ , which is obvious from (267).
- 3) Multiple  $L$ -values for a given  $(k, k_2)$  occur only when  $k_2 \geq 2$  in which case the additional label  $w$  in (177) must be used to distinguish them. Bargmann & Moshinsky found the operator corresponding to this label.
- 4) For  $N \leq 4$ ,  $(k, k_2)$ -values appear only once under a given  $[f]$ . To distinguish multiplicities occurring elsewhere, eg.,  $N[f](k, k_2) = 5[32](40)^2$ , additional label  $\alpha$  of (177) must be made available.
- 5) For further convenience, the eigenvalues

$$E^{(k, k_2)} \equiv E_L^{(k, k_2)} + \frac{1}{2} L(L+1) = \frac{2}{3}(k_1+k_2)^2 - 2k_1(k_2-1) \quad (268)$$

of the  $Q^2$  interaction, save for the term  $\frac{1}{2} L(L+1)$ , are given in the last column. The  $SU_3$  representations are listed in order of increasing energy recalling that larger  $E^{(k, k_2)}$  lie lower in energy because  $Q^2$  is attractive. The base truncation suggested by Elliot consists in taking that  $(k, k_2)$  lying lowest in energy and limiting calculation to this representation. 6) Finally, some  $2s-1d$  shell nuclides are written in parenthesis under that  $[f]$  partition which will presumably explain its lowest states. More details are given later on this point.

The  $SU_3$  representation falling lowest in energy (having

maximum Casimir eigenvalue  $\mathcal{E}^{(k_1, k_2)}$  seen always to be given by

$$\begin{aligned} k_1 &\equiv h_{13} - h_{33} = 2(h_1 - h_6) + (h_2 - h_5) \\ k_2 &\equiv h_{23} - h_{33} = 2(h_4 - h_6) + (h_2 - h_3) \end{aligned} \quad (269)$$

where  $[h_1, h_2, h_3, \dots, h_r] \equiv [f]$  refer to the Young partition. These relations follow directly from the  $U_3$  weight operators  $C_1^1, C_2^2$  and  $C_3^3$  (with eigenvalues  $h_{13}, h_{23}$  and  $h_{33}$  listed in (138)). The construction of polynomials transforming irreducibly according to this  $SU_3$  representation will be a simple matter if one uses the prescription of which (79) served as an example and will be illustrated in detail for  $F^{20}$  and other nuclei.

TABLE V.11 (for 2s-1d Shell)

N	$\underline{U_6}$ [f]	$\underline{SU_3}$ ( $k_1, k_2$ )	$\frac{R_3}{L}$	$\mathcal{E}^{(k_1, k_2)}$
1	[1] ( $0^0, F^0$ )	(20)	0, 2	$6\frac{2}{3}$
2	[2] ( $0^0, F^0$ )	(40) (22)	0, 2, 4 0, 2	$18\frac{2}{3}$ $6\frac{2}{3}$
		[11] (31)	1, 2, 3	$10\frac{2}{3}$
3	[3] ( $F^{10}, Ne^{10}$ )	(60) (42) (00)	0, 2, 4, 6 $0, (2)^2, 3, 4$ 0	36 16 0
		[21] ( $0^0$ )	1, 2, 3, 4, 5 $0, (2)^2, 3, 4$	24 16
		(21)	1, 2	6

4	[4] (Ne <sup>20</sup> )	(80)	0, 2, 4, 6, 8	58 <sup>2</sup> / <sub>3</sub>
		(62)	0, (2) <sup>2</sup> , 3, (4) <sup>2</sup> , 5, 6	30 <sup>2</sup> / <sub>3</sub>
		(44)	0, 2, 4	18 <sup>2</sup> / <sub>3</sub>
		(20)	0, 2	6 <sup>2</sup> / <sub>3</sub>
	[31] (F <sup>20</sup> , Na <sup>20</sup> )	(71)	1, 2, 3, 4, 5, 6, 7	42 <sup>2</sup> / <sub>3</sub>
		(62)	0, (2) <sup>2</sup> , 3, (4) <sup>2</sup> , 5, 6	30 <sup>2</sup> / <sub>3</sub>
		(53)	1, 2, (3) <sup>2</sup> , 4, 5	22 <sup>2</sup> / <sub>3</sub>
		(41)	1, 2, 3, 4	16 <sup>2</sup> / <sub>3</sub>
		(32)	1, 2, 3	10 <sup>2</sup> / <sub>3</sub>
		(20)	0, 2	6 <sup>2</sup> / <sub>3</sub>
	[22] (O <sup>20</sup> )	(62)	0, (2) <sup>2</sup> , 3, (4) <sup>2</sup> , 5, 6	30 <sup>2</sup> / <sub>3</sub>
		(44)	0, 2, 4	18 <sup>2</sup> / <sub>3</sub>
		(41)	1, 2, 3, 4	16 <sup>2</sup> / <sub>3</sub>
		(20)	0, 2	6 <sup>2</sup> / <sub>3</sub>
5	[41] (Ne <sup>21</sup> , Na <sup>21</sup> )	(91)	1, 2, 3, 4, 5, 6, 7, 8, 9	66 <sup>2</sup> / <sub>3</sub>
		(82)	0, (2) <sup>2</sup> , 3, (4) <sup>2</sup> , 5, (6) <sup>2</sup> , 7, 8	50 <sup>2</sup> / <sub>3</sub>
		(73)	1, 2, (3) <sup>2</sup> , (4) <sup>2</sup> , (5) <sup>2</sup> , 6, 7	38 <sup>2</sup> / <sub>3</sub>
		(61)	1, 2, 3, 4, 5, 6	32 <sup>2</sup> / <sub>3</sub>
		(64)	0, (2) <sup>2</sup> , 3, (4) <sup>2</sup> , 5, 6	30 <sup>2</sup> / <sub>3</sub>
		(52)	1, 2, (3) <sup>2</sup> , 4, 5	22 <sup>2</sup> / <sub>3</sub>
		(40)	0, 2, 4	18 <sup>2</sup> / <sub>3</sub>
		(43)	1, 2, 3, 4	16 <sup>2</sup> / <sub>3</sub>
		(31)	1, 2, 3	10 <sup>2</sup> / <sub>3</sub>
		(22)	0, 2	6 <sup>2</sup> / <sub>3</sub>
		(82)	0, (2) <sup>2</sup> , 3, (4) <sup>2</sup> , 5, (6) <sup>2</sup> , 7, 8	50 <sup>2</sup> / <sub>3</sub>
		(73)	1, 2, (3) <sup>2</sup> , (4) <sup>2</sup> , (5) <sup>2</sup> , 6, 7	38 <sup>2</sup> / <sub>3</sub>
		(61)	1, 2, 3, 4, 5, 6	32 <sup>2</sup> / <sub>3</sub>
		(64)	0, (2) <sup>2</sup> , 3, (4) <sup>2</sup> , 5, 6	30 <sup>2</sup> / <sub>3</sub>
		(52)	1, 2, (3) <sup>2</sup> , 4, 5	22 <sup>2</sup> / <sub>3</sub>
		(40) <sup>2</sup>	0, 2, 4 for each (40).	18 <sup>2</sup> / <sub>3</sub>
		(43)	1, 2, 3, 4	16 <sup>2</sup> / <sub>3</sub>
		(31)	1, 2, 3	10 <sup>2</sup> / <sub>3</sub>
		(22)	0, 2	6 <sup>2</sup> / <sub>3</sub>

6	[42] (Ne <sup>22</sup> , Na <sup>22</sup> )	(10,2)	0,2 <sup>2</sup> , 3, (4) <sup>2</sup> , 5, (6) <sup>2</sup> , 7, (8) <sup>2</sup> , 9, 10	76
		(9,3)	1, 2, (3) <sup>2</sup> , (4) <sup>2</sup> , (5) <sup>2</sup> , (6) <sup>2</sup> , (7) <sup>2</sup> , 8, 9	60
		(8,4) <sup>2</sup>	0,2 <sup>2</sup> , 3, (4) <sup>2</sup> , (5) <sup>2</sup> , (6) <sup>2</sup> , 7, 8 for each (9)	48
		⋮		
	[411]	(9,3)	1, 2, (3) <sup>2</sup> , (4) <sup>2</sup> , (5) <sup>2</sup> , (6) <sup>2</sup> , (7) <sup>2</sup> , 8, 9	60
		(7,5)	1, 2, (3) <sup>2</sup> , 4, (5) <sup>2</sup> , 6, 7	40
		(7,2)	1, 2, (3) <sup>2</sup> , 4, (5) <sup>2</sup> , 6, 7	40
		⋮		
7	[43] (Na <sup>23</sup> , Mg <sup>23</sup> )	(11,3)	1, 2, (3) <sup>2</sup> , (4) <sup>2</sup> , (5) <sup>2</sup> , (6) <sup>2</sup> , (7) <sup>2</sup> , (8) <sup>2</sup> , (9) <sup>2</sup> 10, 11	86 <sup>2</sup> / <sub>3</sub>
		(10,4)	0,2 <sup>2</sup> , 3, (4) <sup>2</sup> , (5) <sup>2</sup> , (6) <sup>2</sup> , (7) <sup>2</sup> , (8) <sup>2</sup> , 9, 10	70 <sup>2</sup> / <sub>3</sub>
		(9,5)	1, 2, (3) <sup>2</sup> , (4) <sup>2</sup> , (5) <sup>2</sup> , (6) <sup>2</sup> , (7) <sup>2</sup> , 8, 9	58 <sup>2</sup> / <sub>3</sub>
		⋮		
	[421] (Ne <sup>23</sup> )	(10,1)	1, 2, 3, 4, 5, 6, 7, 8, 9, 10	80 <sup>2</sup> / <sub>3</sub>
		(10,4)	0,2 <sup>2</sup> , 3, (4) <sup>2</sup> , (5) <sup>2</sup> , (6) <sup>2</sup> , (7) <sup>2</sup> , (8) <sup>2</sup> , 9, 10	70 <sup>2</sup> / <sub>3</sub>
		(9,2) <sup>2</sup>	1, 2, (3) <sup>2</sup> , 4, (5) <sup>2</sup> , 6, (7) <sup>2</sup> , 8, 9 for each (9)	62 <sup>2</sup> / <sub>3</sub>
		(9,5)	1, 2, (3) <sup>2</sup> , (4) <sup>2</sup> , (5) <sup>2</sup> , (6) <sup>2</sup> , (7) <sup>2</sup> , 8, 9	58 <sup>2</sup> / <sub>3</sub>
		⋮		
8	[44] (Mg <sup>24</sup> )	(12,4)	0,2 <sup>2</sup> , 3, 4 <sup>2</sup> , 5 <sup>2</sup> , 6 <sup>2</sup> , 7 <sup>2</sup> , 8 <sup>2</sup> , 9 <sup>2</sup> , 10 <sup>2</sup> , 11, 12	98 <sup>2</sup> / <sub>3</sub>
		(10,3)	1, 2, 3 <sup>2</sup> , 4 <sup>2</sup> , 5 <sup>2</sup> , 6 <sup>2</sup> , 7 <sup>2</sup> , 8 <sup>2</sup> , 9, 10	72 <sup>2</sup> / <sub>3</sub>
		(10,6)	0,2 <sup>2</sup> , 3, 4 <sup>2</sup> , 5 <sup>2</sup> , 6 <sup>2</sup> , 7 <sup>2</sup> , 8 <sup>2</sup> , 9, 10	70 <sup>2</sup> / <sub>3</sub>
		⋮		
	[431] (Na <sup>24</sup> , Al <sup>24</sup> )	(11,2)	1, 2, 3 <sup>2</sup> , 4, 5 <sup>2</sup> , 6, 7 <sup>2</sup> , 8, 9 <sup>2</sup> , 10, 11	90 <sup>2</sup> / <sub>3</sub>
		(11,5)	1, 2, 3 <sup>2</sup> , 4 <sup>2</sup> , 5 <sup>2</sup> , 6 <sup>2</sup> , 7 <sup>2</sup> , 8 <sup>2</sup> , 9 <sup>2</sup> , 10, 11	82 <sup>2</sup> / <sub>3</sub>
		(10,3) <sup>2</sup>	1, 2, 3 <sup>2</sup> , 4 <sup>2</sup> , 5 <sup>2</sup> , 6 <sup>2</sup> , 7 <sup>2</sup> , 8 <sup>2</sup> , 9, 10	72 <sup>2</sup> / <sub>3</sub>
		(10,6)	0,2 <sup>2</sup> , 3, 4 <sup>2</sup> , 5 <sup>2</sup> , 6 <sup>2</sup> , 7 <sup>2</sup> , 8 <sup>2</sup> , 9, 10	70 <sup>2</sup> / <sub>3</sub>
		⋮		

9	[441] (Mg <sup>25</sup> , Al <sup>25</sup> )	(12, 8)	1, 2, 3 <sup>2</sup> , 4 <sup>2</sup> , 5 <sup>2</sup> , 6 <sup>2</sup> , 7 <sup>2</sup> , 8 <sup>2</sup> , 9 <sup>2</sup> , 10 <sup>2</sup> , 11, 12	102
		(12, 6)	0, 2 <sup>2</sup> , 4 <sup>2</sup> , 6 <sup>2</sup> , 8 <sup>2</sup> , 10 <sup>2</sup> , 12	96
		(11, 4)	1, 2, 3 <sup>2</sup> , 4 <sup>2</sup> , 5 <sup>2</sup> , 6 <sup>2</sup> , 7 <sup>2</sup> , 8 <sup>2</sup> , 9 <sup>2</sup> , 10, 11	84
		⋮		
	[432] (Na <sup>25</sup> )	(11, 1)	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11	96
		(11, 7)	1, 2, 3 <sup>2</sup> , 4 <sup>2</sup> , 5 <sup>2</sup> , 6 <sup>2</sup> , 7 <sup>2</sup> , 8 <sup>2</sup> , 9 <sup>2</sup> , 10, 11	84
		(11, 4)	1, 2, 3 <sup>2</sup> , 4 <sup>2</sup> , 5 <sup>2</sup> , 6 <sup>2</sup> , 7 <sup>2</sup> , 8 <sup>2</sup> , 9 <sup>2</sup> , 10, 11	84
		⋮		
10	[442] (Mg <sup>26</sup> , Al <sup>26</sup> )	(12, 2)	0, 2 <sup>2</sup> , 3, 4 <sup>2</sup> , 5, 6 <sup>2</sup> , 7, 8 <sup>2</sup> , 9, 10 <sup>2</sup> , 11, 12	106 <sup>2</sup> / <sub>3</sub>
		(12, 8)	0, 2 <sup>2</sup> , 3, 4 <sup>2</sup> , 5 <sup>2</sup> , 6 <sup>2</sup> , 7 <sup>2</sup> , 8 <sup>2</sup> , 9 <sup>2</sup> , 10 <sup>2</sup> , 11, 12	98 <sup>2</sup> / <sub>3</sub>
		(12, 5)	1, 2, 3 <sup>2</sup> , 4 <sup>2</sup> , 5 <sup>2</sup> , 6 <sup>2</sup> , 7 <sup>2</sup> , 8 <sup>2</sup> , 9 <sup>2</sup> , 10 <sup>2</sup> , 11, 12	96 <sup>2</sup> / <sub>3</sub>
		⋮		
	[4411]	(12, 5)	1, 2, 3 <sup>2</sup> , 4 <sup>2</sup> , 5 <sup>2</sup> , 6 <sup>2</sup> , 8 <sup>2</sup> , 9 <sup>2</sup> , 10 <sup>2</sup> , 11, 12	96 <sup>2</sup> / <sub>3</sub>
		(11, 3)	1, 2, 3 <sup>2</sup> , 4 <sup>2</sup> , 5 <sup>2</sup> , 6 <sup>2</sup> , 7 <sup>2</sup> , 8 <sup>2</sup> , 9 <sup>2</sup> , 10, 11	86 <sup>2</sup> / <sub>3</sub>
		(11, 6)	1, 2, 3 <sup>2</sup> , 4 <sup>2</sup> , 5 <sup>2</sup> , 6 <sup>2</sup> , 7 <sup>2</sup> , 8 <sup>2</sup> , 9 <sup>2</sup> , 10, 11	82 <sup>2</sup> / <sub>3</sub>
		⋮		
11	[443] (Mg <sup>27</sup> )	(12, 1)	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	112 <sup>2</sup> / <sub>3</sub>
		(12, 10)	0, 2 <sup>2</sup> , 3, 4 <sup>2</sup> , 5, 6 <sup>2</sup> , 7, 8 <sup>2</sup> , 9, 10 <sup>2</sup> , 11, 12	106 <sup>2</sup> / <sub>3</sub>
		(12, 4)	0, 2 <sup>2</sup> , 3, 4 <sup>2</sup> , 5 <sup>2</sup> , 6 <sup>2</sup> , 7 <sup>2</sup> , 8 <sup>2</sup> , 9 <sup>2</sup> , 10 <sup>2</sup> , 11, 12	98 <sup>2</sup> / <sub>3</sub>
		(12, 7)	2, 3, 4 <sup>2</sup> , 5 <sup>2</sup> , 6 <sup>2</sup> , 7 <sup>2</sup> , 8 <sup>2</sup> , 9 <sup>2</sup> , 10 <sup>2</sup> , 11, 12	96 <sup>2</sup> / <sub>3</sub>
	[4421] (Al <sup>27</sup> , Si <sup>27</sup> )	(12, 4)	0, 2 <sup>2</sup> , 3, 4 <sup>2</sup> , 5 <sup>2</sup> , 6 <sup>2</sup> , 7 <sup>2</sup> , 8 <sup>2</sup> , 9 <sup>2</sup> , 10 <sup>2</sup> , 11, 12	98 <sup>2</sup> / <sub>3</sub>
		(12, 7)	2, 3, 4 <sup>2</sup> , 5 <sup>2</sup> , 6 <sup>2</sup> , 7 <sup>2</sup> , 8 <sup>2</sup> , 9 <sup>2</sup> , 10 <sup>2</sup> , 11, 12	96 <sup>2</sup> / <sub>3</sub>
		(11, 2)	1, 2, 3 <sup>2</sup> , 4, 5 <sup>2</sup> , 6, 7 <sup>2</sup> , 8, 9 <sup>2</sup> , 10, 11	90 <sup>2</sup> / <sub>3</sub>
		⋮		

12	[444] ( $S_4^{24}$ )	(12,0)	0, 2, 4, 6, 8, 10, 12	120
		(12,3)	1, 2, 3 <sup>2</sup> , 4 <sup>2</sup> , 5 <sup>2</sup> , 6 <sup>2</sup> , 7 <sup>2</sup> , 8 <sup>2</sup> , 9 <sup>2</sup> , 10 <sup>2</sup> , 11, 12	102
		(12,6)	0, 2 <sup>2</sup> , 4 <sup>3</sup> , 6 <sup>4</sup> , 8 <sup>3</sup> , 10 <sup>2</sup> , 12	96
		⋮		
	[4431] ( $A_4^{24}$ )	(12,3)	1, 2, 3 <sup>2</sup> , 4 <sup>2</sup> , 5 <sup>2</sup> , 6 <sup>2</sup> , 7 <sup>2</sup> , 8 <sup>2</sup> , 9 <sup>2</sup> , 10 <sup>2</sup> , 11, 12	102
		(12,9)	1, 2, 3 <sup>2</sup> , 4 <sup>2</sup> , 5 <sup>2</sup> , 6 <sup>2</sup> , 7 <sup>2</sup> , 8 <sup>2</sup> , 9 <sup>2</sup> , 10 <sup>2</sup> , 11, 12	102
		(12,6)	0, 2 <sup>2</sup> , 4 <sup>3</sup> , 6 <sup>4</sup> , 8 <sup>3</sup> , 10 <sup>2</sup> , 12	96
		⋮		

## 2. The Pairing ( $R_r$ group) Scheme.

It was shown in (IV, 3. b.) that the mixed-orbital pairing operator  $\mathcal{P}$  given in (219) and (225) ---- which approximates the short-range character of our central two - body interaction ---- is invariant in the chain  $\mathcal{U}_r \supset R_r$ , where as usual  $r$  is the  $\nu$ -shell orbital degeneracy  $r = \frac{1}{2}(\nu+1)(\nu+2)$ . The spectrum of  $\mathcal{P}$  in a base transforming irreducibly according to  $\mathcal{U}_r \supset R_r$  is given by (229 a) and is a function only of irreducible representations  $[f] \equiv [h, h_1 \dots h_r]$  and  $(\lambda, \lambda_1 \dots \lambda_r)$  of  $\mathcal{U}_r$  and  $R_r$  respectively. The number  $K$  of labels required to specify an irreducible representation of  $R_r$  is given in terms of  $r$  by (228). One can further label the states by orbital angular momentum  $L$  and projection  $M_L$  by reducing  $R_r$  explicitly according to  $R_3 \supset R_2$  as in chain (229 b) corresponding to kets (229c.) Additional symmetries (Pauli exclusion, spin and isopin) can be incorporated into the base which transforms irreducibly

according to the chain

$$\begin{array}{ccccccccccc}
 U_{4r} & \supset & U_r \times U_4 & \supset & R_r \times U_4 & \supset & R_3 \times U_4 & \supset & R_3 \times U_2 \times U_2 & \supset & R_2(o) \times R_2(s) \times R_2(i) \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 [1^N 0^{4r-N}] & & [f] & & \alpha'(\lambda_1 \lambda_2 \dots \lambda_k) & & \beta \omega' L & & S & & T & & M_L & & M_S & & M_T
 \end{array}
 \quad (270)$$

$$k = \frac{1}{2}(r-1) \quad \underline{r \text{ odd}} ; \quad k = \frac{1}{2}r \quad \underline{r \text{ even}}$$

where  $\alpha', \beta$  and  $\omega'$  respectively designate eigenvalues of operators required to distinguish multiplicities in the reductions.  $U_r \supset R_r$ ,  $U_4 \supset U_2 \times U_2$  and  $R_r \supset R_3$ , when and if they are needed. (This chain of groups under which  $P$  is diagonal should be compared with chains (99) and (195a) under which  $Q^2$  is diagonal.

Calculations in this scheme would thus require the construction of the set of totally anti-symmetric states

$$[f] \alpha'(\lambda_1 \lambda_2 \dots \lambda_k) \beta \omega' L S T, M_L M_S M_T \rangle. \quad (271)$$

The irreducible representations  $(\lambda_1 \lambda_2 \dots \lambda_k)$  of  $R_r$  contained in a given one  $[f] \equiv [h_1 h_2 \dots h_r]$  of  $U_r$  can be derived by a technique given by Jahn<sup>41)</sup> (1950). The L-structure of each  $R_r$  representation is in turn found by chain calculations of the type illustrated before. Elliot<sup>12)</sup>

provides a table for the more symmetric Young partitions in the 2s-1d shell ( $r=6$ ) for  $N \leq 4$  particles and we here reproduced that table with the pairing force eigenvalues (229 a).

according to the chain

$$\begin{array}{ccccccccccc}
 U_{4r} & \supset & U_r \times U_4 & \supset & R_r \times U_4 & \supset & R_3 \times U_4 & \supset & R_3 \times U_2 \times U_2 & \supset & R_2(o) \times R_2(s) \times R_2(i) \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
 [1^N 0^{4r-N}] & & [f] & & \alpha'(\lambda, \lambda_2 \dots \lambda_K) & & \beta \omega' L & S & T & M_L & M_S & M_T
 \end{array}
 \quad (270)$$

$$K = \frac{1}{2}(r-1) \quad \underline{\text{r odd}} ; \quad K = \frac{1}{2}r \quad \underline{\text{r even}}$$

where  $\alpha', \beta$  and  $\omega'$  respectively designate eigenvalues of operators required to distinguish multiplicities in the reductions.  $U_r \supset R_r$ ,  $U_4 \supset U_2 \times U_2$  and  $R_r \supset R_3$ , when and if they are needed. (This chain of groups under which  $P$  is diagonal should be compared with chains (99) and (145a) under which  $Q^2$  is diagonal.

Calculations in this scheme would thus require the construction of the set of totally anti-symmetric states

$$[f] \alpha'(\lambda, \lambda_2 \dots \lambda_K) \beta \omega' L S T, M_L M_S M_T \rangle. \quad (271)$$

The irreducible representations  $(\lambda, \lambda_2 \dots \lambda_K)$  of  $R_r$  contained in a given one  $[f] \equiv [h, h_2 \dots h_r]$  of  $U_r$  can be derived by a technique given by Jahn<sup>41)</sup> (1950). The L-structure of each  $R_r$  representation is in turn found by chain calculations of the type illustrated before. Elliot<sup>12)</sup> provides a table for the more symmetric Young partitions in the 2s-1d shell ( $r=6$ ) for  $N \leq 4$  particles and we here reproduced that table with the pairing force eigenvalues (229 a).



More pronounced degeneracy in the  $L$ -structure is present here than in the case  $u_6 > u_3 > R_3$ .

Typical example: The 3 particle partition  $[21]$  contains  $L = 0, 1^2, 2^4, 3^2, 4^2, 5$  which in the  $Q^2$  scheme is broken into three levels (see Table V.1.1 ) while only two levels result under the  $P$  scheme (Table V. 2.1.).

Multiple  $(\lambda, \lambda_2, \lambda_3)$  of a given  $[f]$  occur only for  $N \geq 6$  --- the label  $\alpha'$  is then needed.

The problem of constructing pairing scheme states (271) is being studied by Chacón at the University of Mexico.

TABLE V.2.1 (for 2s-1d Shell)

N	$\underline{U}_6$ [f]	$\underline{R}_6$ ( $\lambda_1 \lambda_2 \lambda_3$ )	$\underline{R}_3$ L	$E_{(\lambda)}^{(f)}$
1	[1]	(100)	0, 2	0
2	[2]	(000)	0	6
		(200)	0, 2 <sup>2</sup> , 4	0
	[11]	(110)	1, 2, 3	0
3	[3]	(100)	0, 2	8
		(300)	0 <sup>2</sup> , 2 <sup>2</sup> , 3, 4 <sup>2</sup> , 6	0
	[21]	(100)	0, 2	4
		(210)	1 <sup>2</sup> , 2 <sup>3</sup> , 3 <sup>2</sup> , 4 <sup>2</sup> , 5	0
4	[4]	(000)	0	16
		(200)	0, 2 <sup>2</sup> , 4	10
		(400)	0 <sup>2</sup> , 2 <sup>3</sup> , 3, 4 <sup>3</sup> , 5, 6 <sup>2</sup> , 8	0
	[31]	(110)	1, 2, 3	8
		(200)	0, 2 <sup>2</sup> , 4	6
		(310)	0, 1 <sup>3</sup> , 2 <sup>4</sup> , 3 <sup>5</sup> , 6 <sup>2</sup> , 7 4 <sup>4</sup> , 5 <sup>3</sup>	0

More pronounced degeneracy in the  $L$ -structure is present here than in the case  $u_6 > u_3 > R_3$ .

Typical example: The 3 particle partition  $[21]$  contains  $L = 0, 1^2, 2^4, 3^2, 4^2, 5$  which in the  $Q^2$  scheme is broken into three levels (see Table V.1.1 ) while only two levels result under the  $P$  scheme (Table V. 2.1.).

Multiple  $(\lambda, \lambda_1, \lambda_2)$  of a given  $[f]$  occur only for  $N \geq 6$  --- the label  $\alpha'$  is then needed.

The problem of constructing pairing scheme states (271) is being studied by Chacón at the University of Mexico.

### 3. The $j$ - $j$ Many - Particle Coupling Scheme Using Unitary Groups.

If the single-particle quantum numbers  $p$  were to be given as  $(\nu l j m, \tau)$  of (15) instead of  $(\nu l m, \sigma \tau)$  of (14) or, with more explicit reference to the oscillator, by  $(n, n_r n_o, \sigma \tau)$  as we have done in our work, then our  $N$  - particle functions of the type (79) would be eigenfunctions of the single - body spin - orbit operator for  $N$  particles

$$\sum_{i=1}^N \vec{l}_i \cdot \vec{A}_i = \frac{1}{2} \sum_i (\vec{j}_i^2 - \vec{l}_i^2 - \vec{A}_i^2)$$

with eigenvalues given by

$$\frac{1}{2} \sum_{i=1}^N [j_i(j_i+1) - l_i(l_i+1) - 3/4]. \quad (272)$$

What would be the chain of groups under which the corresponding set of eigenfunctions transform irreducibly?

The largest group of transformations is of course  $U_{4r}$ ,  $r$  being the orbital degeneracy involved and 4 denoting the spin - isospin degeneracy. If the major oscillator shell contains but one sub-shell, i.e., a single  $j$  value, then one may certainly consider the whole space as broken up into spin-orbital and isospin subspaces (of dimensions  $2j+1$  and 2, respectively) leading to the chain

$$U_{4r} \equiv U_{2(2j+1)} \supset U_{2j+1} \times U_2 \supset R_3 (\text{spin-orbital}) \times U_2 (\text{isospin}).$$

This chain is easily generalized to the case of  $K$  sub-shells within our given mayor shell by considering the "direct sum" of unitary groups  $U_{2j_i+1}$  where  $i=1, 2, \dots, K$ :

$$U_{4r} \supset U_{2r} \times U_2 \supset \begin{pmatrix} u_{2j_1+1} & & 0 \\ & u_{2j_2+1} & \\ 0 & \ddots & u_{2j_k+1} \end{pmatrix} \times U_2 \supset R_3 \times U_2 \quad (273a)$$

where our notation means the direct product

$$\begin{pmatrix} u_{2j_1+1} & & 0 \\ & u_{2j_2+1} & \\ 0 & \ddots & u_{2j_k+1} \end{pmatrix} \equiv \begin{pmatrix} u_{2j_1+1} & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix} \times \begin{pmatrix} 1 & & \\ & u_{2j_1+1} & \\ & & \ddots \\ & & & 1 \end{pmatrix} \times \dots \times \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \\ & & & u_{2j_k+1} \end{pmatrix} \quad (273b)$$

and  $R_3$  refers to rotations in the full spin - orbital sub-space, i.e., in representation language, to

$$R_3 \Rightarrow \begin{pmatrix} D^{(j_1)} & & 0 \\ & D^{(j_2)} & \\ 0 & \ddots & D^{(j_k)} \end{pmatrix} \quad (273c)$$

while  $U_2$  refers to isospinorial space. The total spin - orbital degeneracy, is simply  $2r$  where

$$2r = \sum_{i=1}^k (2j_i + 1).$$

In the  $2s-1d$  shell there are 3 sub-shells, namely,  $d_{5/2}$ ,  $s_{1/2}$  and  $d_{3/2}$  (see Figure V.III.6.1) and the chain involved is thus:

$$\begin{array}{ccccccc}
 U_{24} & \supset & U_{12} \times U_2 & \supset & \begin{pmatrix} U_6 & & 0 \\ & U_2 & \\ 0 & & U_4 \end{pmatrix} & \times & U_2 & \supset & R_3 \times U_2 & (274) \\
 \uparrow & & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
 [1^N 0^{24-N}] & & [g] & T & [g'] & [g''] & [g'''] & T & J \quad T
 \end{array}$$

in accordance with (273a, b, c,) the irreducible representation labels (quantum numbers) of the various groups being given below them. Notice that, as desired, our totally antisymmetric N - particle states would be of definite J and T.

Consider a single particle in the  $2s - 1d$  shell. The  $U_{12}$  Young diagram  $[g]$  is simply  $[1]$  and its dual representation  $[\tilde{1}] = [1]$  is the corresponding diagram for  $U_2$  (isospin) required by the exclusion principle----thus,  $T = 1/2$ . The particle can be in either of the three sub-shells  $d_{5/2}$ ,  $s_{1/2}$  or  $d_{3/2}$  so that there are three irreducible representations, each given by  $[g'] [g''] [g'''] \equiv \{g', g'', g'''\}$ , of the subgroup

$$\begin{pmatrix} U_6 & & 0 \\ & U_2 & \\ 0 & & U_4 \end{pmatrix} \quad (275)$$

contained in the  $[g] = [1]$  representation of  $U_{12}$ , namely,

$$\begin{aligned}
 \{1, 0, 0\} & \supset J = j_1 = 5/2 \\
 \{0, 1, 0\} & \supset J = j_2 = 1/2 \\
 \{0, 0, 1\} & \supset J = j_3 = 3/2
 \end{aligned} \quad (276)$$

The first, second and third numbers in the brackets refer to the Young diagrams, respectively, of  $U_1$ ,  $U_2$  and  $U_3$ .

The irreducible representations of  $R_3$  (J-structure) are shown to the right of each bracket.

Which are the irreducible representations of subgroup (275) contained in a representation  $[g] = [1^N 0^{12-N}]$ , that is, totally antisymmetric, of  $U_{12}$ ? The 12 single particle states ( $\nu l f m_f$ ) referring to spin - orbital coordinates for the  $2s-1d$  shell can be labelled with  $\mu$  by, say,

$\mu$	1	2	3	4	5	6	7	8	9	10	11	12
$l$	2	2	2	2	2	2	0	0	2	2	2	2
$f$	$5/2$	$5/2$	$5/2$	$5/2$	$5/2$	$5/2$	$1/2$	$1/2$	$3/2$	$3/2$	$3/2$	$3/2$
$m_f$	$5/2$	$3/2$	$1/2$	$-1/2$	$-3/2$	$-5/2$	$1/2$	$-1/2$	$3/2$	$1/2$	$-1/2$	$-3/2$
	$d_{5/2}$						$s_{1/2}$		$d_{3/2}$			

so that the generators of  $U_{12}$  are  $b_{\mu}^{\mu'} \equiv b_{l f m_f}^{l f m_f}$  ( $\mu, \mu' = 1, 2, \dots, 12$ ) and obey commutation relations like (35). An  $N$  - particle basis irreducible under  $U_{12}$  and also explicitly under subgroup (275) is the general polynomial

$$P_{\{g', g'', g'''\}}^{[g]} |0\rangle \equiv | [g] \{g', g'', g'''\} \rangle = \sum_{\mu_1, \mu_2, \dots, \mu_N} A_{\mu_1, \mu_2, \dots, \mu_N} b_{\mu_1}^+ b_{\mu_2}^+ \dots b_{\mu_N}^+ |0\rangle$$

of degree  $N$ . In accordance with (73), to find the

$\{g', g'', g'''\}$  contained in  $[g] = [1^N 0^{12-N}]$  one must require the conditions on  $P$  that

$$1) \ell_{\mu}^{\mu} P = g_{\mu} P \quad \begin{cases} [g'] \equiv [g_1, g_2, \dots, g_6] \\ [g''] \equiv [g_1, g_7] \\ [g'''] \equiv [g_1, g_{10}, g_{11}, g_{12}] \end{cases}$$

$$g_{\mu} = 0 \text{ or } 1$$

$$g_1 \geq g_2 \geq \dots \geq g_6; \quad g_7 \geq g_8; \quad g_9 \geq g_{10} \geq g_{11} \geq g_{12}$$

$$2) \ell_{\mu}^{\mu'} P = 0 \quad \begin{cases} 1 \leq \mu < \mu' \leq 6 \\ 7 \leq \mu < \mu' \leq 8 \\ 9 \leq \mu < \mu' \leq 12 \end{cases}$$

so that  $P$  is of maximum weight in  $\mathcal{U}_{12}$  and in subgroup (275).  
A polynomial of arbitrary weight for, say  $N = 5$ , would be labelled by say

$$\{ 110100, 10, 0100 \} \quad N = 5$$

but to be of maximum weight it would have to be designated by

$$\{ 111000, 10, 1000 \}$$

in accordance with (1) above. Our polynomial would then have to contain the factors  $b_1^+, b_2^+, b_3^+, b_7^+$  and  $b_9^+$ ; in fact the polynomial is simply

$$b_1^+ b_2^+ b_3^+ b_7^+ b_9^+$$

up to a multiplicative constant for normalization. A similar analysis would give us the other  $\{g', g'', g'''\}$  for  $[g] = [11111]$  apart from the  $\{111, 1, 1\}$  already obtained.



For  $N=2$  consider the  $[11]$  partition of  $\mathcal{U}_{12}$  which has the dual partition  $[\tilde{11}] = [2]$  of  $\mathcal{U}_2$  implying that  $T=1$ . The possible representations of subgroups (275) involved here are then:

$$[11] \supset \{11, 0, 0\} \{1, 1, 0\} \{1, 0, 1\} \{0, 11, 0\} \{0, 1, 1\} \text{ and } \{0, 0, 11\} \quad (278)$$

each possessing a definite  $J$  - structure. To find the  $\{g', g'', g'''\}$  contained in the  $[g] = [2], [\tilde{g}] = [11] \Rightarrow T=0$  case, we take the external product of  $[1]$  with  $[1]$  in  $\mathcal{U}_{12}$  so that

$$[1] \otimes [1] \equiv \square \otimes \square = \square + \square$$

according to Littlewood's rules. Taking all the possible external products of (276) among themselves we find that

$$\mathcal{U}_{12} \supset \begin{pmatrix} \mathcal{U}_6 & & 0 \\ & \mathcal{U}_2 & \\ 0 & & \mathcal{U}_4 \end{pmatrix}$$

$$\square \otimes \square = \square + \square \supset \{2, 0, 0\} \{11, 0, 0\} \{1, 1, 0\} \{1, 0, 1\} \{1, 1, 0\} \{0, 2, 0\} \\ \{0, 11, 0\} \{0, 1, 1\} \{1, 0, 1\} \{0, 1, 1\} \{0, 0, 2\} \{0, 0, 11\},$$

and subtracting (278) from this, one is left with:

$$\square \supset \{2, 0, 0\} \{1, 1, 0\} \{0, 2, 0\} \{1, 0, 1\} \{0, 1, 1\} \{0, 0, 2\}.$$

Now,  $\{2,0,0\}$  means 2 particles arranged symmetrically in the  $d_{5/2}$  sub-shell, the possible  $J$  - values are thus  $J=1,3,5$ . Representation  $\{1,1,0\}$  means a configuration  $(d_{5/2})^1 (s_{1/2})^1$  so that,  $J$  being  $j_1 + j_2$  vectorially, we have  $J=2,3$ . Then  $\{0,2,0\}$  contains simply  $J=1$ , and so forth. Tabulating these typical results for  $[g] = [2]$  we have

TABLE V.3.1.

$u_{12}$	$U_2$	$(u_1 u_2 u_4)$	$J$	$D$
[2]	T=0	$\{2,0,0\}$	1,3,5	21
		$\{1,1,0\}$	2,3	12
		$\{0,2,0\}$	1	3
		$\{1,0,1\}$	1,2,3,4	24
		$\{0,1,1\}$	1,2	8
		$\{0,0,2\}$	1,3	10

$$\sum D = 78$$

where the numbers in the last column refer to the dimensionality (c.f., Hammermesh, p. 387, ref. 34) of the representations involved. The  $D$ 's are given in each case in terms of the Young partition numbers of the unitary group involved (Weyl dimensionality formula) and are the analogue of  $(2J+1)$  for an  $R_3$  representation given by  $J$ . Their use provides a good check on the  $\{g', g'', g'''\}$  's obtained from a given  $[g]$ . For example, in the above Table: for  $\{2,0,0\} \supset J=1,3,5$  and for which  $D = 21$ , one must have (recalling definition (273b))

$$D(u_1) D(u_2) D(u_4) = \sum_{J=1,3,5} (2J+1)$$

$$2! \times 1! \times 1! = 3+7+11$$

$$21 = 21.$$

Again, the grand total of 78 must result from

$$D(\mathcal{U}_{12}) = \sum_{\text{all } J} D(\mathcal{U}_6) D(\mathcal{U}_2) D(\mathcal{U}_4) = \sum_{\text{all } J} (2J+1) = 78$$

for the  $[g]=[2]$  representation of  $\mathcal{U}_{12}$ .

To continue our chain calculation, the  $\mathcal{U}_{12}$  representation  $[111]$ , with conjugate  $U_2$  representation  $[\tilde{111}] = [3] \Rightarrow T = 3/2$ , contains the following  $\{g', g'', g'''\}$  of subgroup( 275):

$$\{11,0,0\} \quad \{11,1,0\} \quad \{11,0,1\} \quad \{1,11,0\} \quad \{1,1,1\} \quad \{1,0,11\}$$

$$\{0,11,1\} \quad \{0,1,11\} \quad \text{and} \quad \{0,0,11\}.$$

By a process identical to the one for deriving the structure of  $[g] = [2]$  in the previous example we can find the structure of  $[g] = [21]$  with  $T = 1/2$  from the structure of  $\mathbb{B} \otimes \square = \mathbb{B} + \mathbb{B}$  and of  $[111]$  above.

The process is repeated for any desired number  $N$  of particles and results like Table V.3.2 tabulated for any wanted  $[g]$  representation. In particular, we shall give the complete table for the case of  $F^{20}$  ( $N = 4$ ) whose lowest levels are given by  $T = 1$  where one has  $[g] = [211]$  and  $[g] = [31]$  which implies  $T_1 = 1$ . We included the triplet of numbers  $(n_1, n_2, n_3)$  giving the distribution of  $N$  particles among the  $2s-1d$  shell according to the configuration  $(d_{5/2})^{n_1} (s_{1/2})^{n_2} (d_{3/2})^{n_3}$ . Also, the eigenvalues (272) corresponding to each such configuration and normalized with respect to the single - particle oxygen - 17 spectrum (c.f., Preston, p. 184, ref. 53) are given in the last column.

TABLE V.3.2.  $U_{12}: [211] (T=1) F^{20}$

$U_6$	$U_2$	$U_4$	$(n_1, n_2, n_3)$	$J$	$D$	NORMALIZED EIGENVALUES OF EQ. (272)
211	0	0	4 0 0	$1^2, 2^2, 3^3, 4^2, 5^2, 6, 7$	105	0 (MeV)
111	1	0	3 1 0	$1, 2^2, 3, 4, 5$	40	.871
21	1	0	3 1 0	$0, 1^2, 2^3, 3^4, 4^3, 5^2, 6, 7$	140	.871
11	2	0	2 2 0	$1^2, 2, 3^2, 4, 5$	45	1.742
111	0	1	3 0 1	$0, 1^2, 2^2, 3^3, 4^2, 5, 6$	80	5.08
21	0	1	3 0 1	$0, 1^4, 2^6, 3^6, 4^6, 5^5, 6^3, 7^2, 8$	280	5.08
11	1	1	2 1 1	$0, 1^3, 2^4, 3^4, 4^3, 5^2, 6$	120	5.951
2	1	1	2 1 1	$0, 1^5, 2^4, 3^4, 4^4, 5^3, 6^2, 7$	168	5.951
1	11	1	1 2 1	$1, 2, 3, 4$	24	6.822
1	2	1	1 2 1	$0, 1^2, 2^3, 3^3, 4^2, 5$	72	6.822
1	1	11	1 1 2	$0, 1^2, 2^3, 3^3, 4^2, 5$	72	11.031
1	1	2	1 1 2	$0, 1^3, 2^4, 3^4, 4^3, 5^2, 6$	120	11.031
11	0	11	2 0 2	$0^2, 1, 2^4, 3^4, 4^3, 5, 6$	90	10.16

11	0	2	202	1 <sup>2</sup> , 2 <sup>3</sup> , 3 <sup>5</sup> , 4 <sup>3</sup> , 5 <sup>3</sup> , 6, 7	150	10.16
1	21	0	130	3, 3	12	2.613
11	11	0	220	0, 2, 4	15	1.742
2	11	0	220	1, 3, 5	21	1.742
11	1	1	211	0, 1 <sup>3</sup> , 2 <sup>4</sup> , 3 <sup>4</sup> , 4 <sup>3</sup> , 5 <sup>2</sup> , 6	120	5.951
2	0	11	202	1 <sup>3</sup> , 2 <sup>2</sup> , 3 <sup>4</sup> , 4 <sup>2</sup> , 5 <sup>3</sup> , 6, 7	126	10.16
1	0	21	103	0, 1 <sup>3</sup> , 2 <sup>4</sup> , 3 <sup>4</sup> , 4 <sup>3</sup> , 5 <sup>2</sup> , 6	120	15.24
1	11	1	121	1, 2, 3, 4	24	6.822
0	21	1	031	1, 2	8	7.693
0	11	2	022	1, 3	10	11.902
1	1	11	112	0, 1 <sup>2</sup> , 2 <sup>3</sup> , 3 <sup>3</sup> , 4 <sup>2</sup> , 5	72	etc.
0	11	11	022	0, 2	6	
0	2	11	022	1 <sup>2</sup> , 3	18	
0	1	21	013	0, 1 <sup>2</sup> , 2 <sup>2</sup> , 3 <sup>2</sup> , 4	40	
1	0	111	103	1, 3, 3, 4	24	
0	1	111	013	1, 2	8	
0	0	211	004	1, 2, 3	15	
$\Sigma D =$					2145	

Conclusion: One would naturally only be interested in the first few lowest configurations (see Table V.3.2) to carry out calculations in the "j-j N - particle coupling scheme" described here. Notice that even these lowest states are very degenerate in J, but it is presumably removed by the  $P$  and  $Q^2$  interactions. Construction of the lowest states of the basis irreducible under the chain of groups(274) has not been attempted as we have decided upon the  $SU_3$  classification scheme for which the construction of states we turn to the following chapter.

# VI. CONSTRUCTION OF MANY - PARTICLE WAVE FUNCTIONS IN THE $SU_3$ SCHEME.

From elementary angular momentum theory we recall that there are two basic methods involved in the construction of an  $N$  - particle basis set of definite total angular momentum  $L$  out of single - particle functions of definite  $l$ . One is a lowering - (or step - ) operator technique and the other utilizes Clebsch - Gordon vector - coupling coefficients. Both will be illustrated briefly as a prelude to their generalization for  $SU_3$ .

The lowering operator technique is essentially contained in (53), from which one can easily see that

$$Y_{LM-1} = \frac{1}{\sqrt{(L+M)(L-M+1)}} L_- Y_{LM} \equiv C_{LM} L_- Y_{LM}. \quad (279)$$

The operator  $L_-$  lowers the value of index  $M$  in steps of one and is given from (279) by

$$L_- = \sum_{i=1}^N L_{-}(i) = \sum_{i=1}^N \left[ x^{(i)} \frac{\partial}{\partial x_+^{(i)}} + x_-^{(i)} \frac{\partial}{\partial x_0^{(i)}} \right]$$

so that if one possessed all the  $Y_{LL}$  functions of maximum weight for a given  $N$  - particle configuration  $(l_1, l_2, \dots, l_N)$

one could by (279) generate the full  $(2L+1)$ -dimensional bases according to the row index  $M$  where  $-L \leq M \leq L$ .

In particular, for maximum  $L$ , namely,  $L = l_1 + l_2 + \dots + l_N$

one has simply  $Y_{LL} = Y_{l_1 l_1} Y_{l_2 l_2} \dots Y_{l_N l_N}$  where from

(51) and (52)  $Y_{l_i l_i} = A_{0i} (x_+^{(i)})^{l_i} \quad (i = 1, 2, \dots, N).$

For non - maximum  $L$ -values the  $Y_{LL}$ 's could be obtained, e.g., by the following procedure for 2 particles:

The weight operation

$$L_0 Y_{l_1 m_1} Y_{l_2 m_2} = M Y_{l_1 m_1} Y_{l_2 m_2}$$

gives that  $M = m_1 + m_2$  as  $L_0 = L_0^{(1)} + L_0^{(2)}$ . A maximum weight (i.e.,  $M = L$ ) two-particle function transforming irreducibly as  $L = l_1 + l_2, l_1 + l_2 - 1, \dots, l_1 - l_2$  is then the linear combination

$$Y_{LL} = \sum_m A_m Y_{l_1 m} Y_{l_2 L-m}$$

where  $m \equiv m_1, m_2 = L - m$ . The coefficients  $A_m$  are easily found through the fact that

$$L_+ Y_{LL} = 0$$

and  $L_+^{(1)} Y_{l_1 m_1} = \sqrt{(l_1 - m_1)(l_1 + m_1 + 1)} Y_{l_1 m_1 + 1}$ .  
One gets then the result

$$\frac{A_{m+1}}{A_m} = - \sqrt{\frac{(l_1 - m)(l_1 + m + 1)}{(l_2 - L + m + 1)(l_2 - L - m)}}$$

allowing us to determine  $A_m$  up to a constant  $A_0$ , that is:

$$Y_{LL} = A_0 \sum_m (-1)^m \sqrt{\frac{(l_1 + m)!(l_2 - L - m)!}{(l_1 - m)!(l_2 - L + m)!}} Y_{l_1 m} Y_{l_2 L-m}.$$

By (279) one could then generate all the  $y_{LM}$  functions for  $L = l_1 + l_2, l_1 + l_2 - 1, \dots, l_1 - l_2$  and  $-L \leq M \leq L$ , in other words: the full basis for irreducible representations  $L$  of the group  $R_3$ .

Alternatively, to build  $\{y_{LM}\}$  for  $N$ - particles one could vector couple the first two particles to definite  $l'm'$  by

$$y_{l'm'} = \sum_{m_1, m_2} \langle l_1, l_2, m_1, m_2 | l'm' \rangle y_{l_1, m_1} y_{l_2, m_2}$$

and couple this to particle 3 to give

$$y_{l''m''} = \sum_{m_3, m'} \langle l_3, l'm_3, m' | l''m'' \rangle y_{l_3, m_3} y_{l'm'},$$

and so on, up to the  $N^{th}$  particle - - - thus obtaining the complete set  $y_{LM}$  for each  $R_3$  irreducible representation  $L$ .

(Note: we disregard the problem of antisymmetrization in our example.)

Both techniques - - - of lowering operators and Clebsch - Gordan coefficients - - - are relatively simple. Coefficients  $\langle l_1, l_2, m_1, m_2 | l'm \rangle$  which refer to the coupling of two  $R_3$  irreducible representations  $l_1$  and  $l_2$  to give a third one  $l$ , have been tabulated extensively. Using the three lowering generators of  $U_3$  and the Wigner coefficients for  $U_3$  which have also been tabulated to some extent, one can generalize for  $U_3$  the above well - known methods.

### 1. Lowering Operator Functions.

From the introductory discussion of (IV,d) we recall that the irreducible representations of  $U_3 \supset U_2 \supset U_1$  are characterized by  $(k_1, k_2, q_1, q_2, r_1)$  through the general bra



$$\underline{R_3 \supset R_2 \text{ ANALOGUE:}} \quad ; \quad \left| \begin{matrix} k, k_2, 0 \\ q, q_2 \\ r_1 \end{matrix} \right\rangle \equiv P_{q, q_2, r_1}^{(k, k_2)} (b_{\mu s}^+) |0\rangle \quad (280)$$

the superscript on the polynomial  $P$  referring to the irreducible representation of  $SU_3$ , the subscript denoting the row indices of that representation. From (235) and (236) moreover, the weight of (280) is given by the three eigenvalues of  $C_1^1, C_2^2$  and  $C_3^3$

$$\begin{aligned} \underline{R_3 \supset R_2 \text{ ANALOGUE:}} \quad ; \quad & w_1 = r_1 \\ & w_2 = (q_1 + q_2) - r_1 \\ & w_3 = (k_1 + k_2) - (q_1 + q_2), \end{aligned} \quad (281)$$

and the polynomial of maximum weight in  $SU_3$  is designated by

$$\underline{R_3 \supset R_2 \text{ ANALOGUE:}} \quad ; \quad \left| \begin{matrix} k, k_2, 0 \\ k, k_2 \\ k_1 \end{matrix} \right\rangle = P_{k, k_2, k_1}^{(k, k_2)} (b_{\mu s}^+) |0\rangle \quad (282)$$

and satisfying (144 a,b), remembering that  $k_1 = h_{13} - h_{33}$  and  $k_2 = h_{23} - h_{33}$ .

The operation in (53) yields an arbitrary  $|lm\rangle \equiv y_{lm}^l$  from the maximum weight  $|ll\rangle = y_{ll}^l(\mu)$  in  $R_3$ . An arbitrary state (280) in  $U_3$  can be generated from (282) by

$$P_{q, q_2, r_1}^{(k, k_2)} = R(C_\sigma^r) P_{k, k_2, k_1}^{(k, k_2)} \quad (283)$$

where  $R(C_\sigma^{\sigma'})$  must be polynomial in the 3 lowering generators of  $U_3$  : namely,

$$R(C_\sigma^{\sigma'}) = \sum_{\alpha\beta\gamma} A_{\alpha\beta\gamma} (C_2^1)^\alpha (C_3^2)^\beta (C_3^1)^\gamma \quad (284)$$

Analogously with (279), it is desirable to have lowering operators which reduce the representation label of a given subgroup in steps of one, keeping the representation in maximum weight with regard to that subgroup. The rows of  $(k, k_2)$  being characterized by three indices  $(q_1, q_2, q_1)$ , there will be three such operators obtainable directly from (284) by imposing on it the appropriate restrictions: The first such operator  $R_I$  is to lower by one the first index  $q_1$  of  $U_2$  maintaining maximum weight in  $U_2$ , i.e.,

$$R_I P_{q_1, q_2, q_1} \equiv P_{q_1-1, q_2, q_1-1}$$

having suppressed the unneeded  $(k, k_2)$  superscripts. Now, from (144) and (281) one can write

$$\begin{aligned} C_1^1 R_I P &= (q_1 - 1) R_I P \\ C_2^2 R_I P &= q_2 R_I P \\ C_1^2 R_I P &= 0 \end{aligned}$$

where  $P \equiv P_{q_1, q_2, q_1}$ , that is, of maximum weight in  $U_2$ . These three equations can be rewritten as

$$[C_1^1, R_I] P + q_1 R_I P = (q_1 - 1) R_I P \quad (a)$$

$$[C_2^2, R_I] P + q_2 R_I P = q_2 R_I P \quad (b)$$

$$[C_1^2, R_I] P = 0. \quad (c)$$

Carrying out these simple commutations using (284) and the fact that from (135), e.g.,

$$[C_1', (C_3')^r] = -r (C_3')^{r-1}$$

then equation (a) gives condition  $\gamma = 1 - \alpha$  and (b) gives  $\beta = \alpha$  so that

$$R_I = \sum_{\alpha} A_{\alpha} (C_2')^{\alpha} (C_3')^{\alpha} (C_3')^{1-\alpha}$$

with the requirement that it stay polynomial ( $\alpha = 0, 1$ ) becomes

$$R_I = A_0 C_3' + A_1 C_2' C_3'^2.$$

Equation (c) then eliminates 1 of the constants so that

$$R_I = A_1 [C_3' (C_1' - C_2'^2 + 1) + C_2' C_3'^2] \quad (285)$$

having used commutation relations (135). Our second operator must lower the second index  $q_2$ ,

$$R_{II} P_{q_1, q_2, q_1} \equiv P_{q_1, q_2-1, q_1}.$$

Exactly the same reasoning as before leads to conditions

$\alpha = \gamma = 0$  and  $\beta = 1$  on (284) leaving

$$R_{II} = A'_1 C_3^2. \quad (286)$$

The third operator  $R_{III}$  is to lower the third index  $r$  by one

$$R_{III} P_{q_1, q_2, r} = P_{q_1, q_2, r-1}$$

and one similarly obtains that  $\alpha = \beta + 1, \beta = r = 0$  so (284) here becomes

$$R_{III} = A''_1 C'_2.$$

The operators  $R_I, R_{II}$  and  $R_{III}$  are precisely the  $U_3, U_2, U_1$  equivalent of the normalized operators  $\mathbb{L}_r^k$  mentioned on page 43 in relation to the canonical chain  $U_r, U_{r-1}, \dots, U_1$ . They have been discussed thoroughly by Nagel & Moshinsky<sup>(45)</sup>, who also obtained the normalization constants for the general case of  $U_r$  by an elegant method relying only on the group and subgroup generator commutation relations. The normalization constants here are  $A_1, A'_1$  and  $A''_1$  and are given in the above reference allowing us to write

$$U_3, U_2 \left\{ \begin{aligned} R_I &\equiv \mathbb{L}_3^1 = \frac{[(q_1 - q_2 + 1) C'_3 + C'_2 C_3^2]}{\sqrt{(q_1 - q_2 + 1)(k_1 - q_1 + 1)(q_1 - k_2)(q_1 + 1)}} \\ R_{II} &\equiv \mathbb{L}_3^2 = \frac{(q_1 - q_2 + 2)}{\sqrt{(k_2 - q_2 + 1)(k_1 - q_2 + 2)q_2}} C_3^2 \end{aligned} \right. \quad (287a, b, c)$$

$$U_2, U_1 \left\{ R_{III} \equiv \mathbb{L}_2^1 = \frac{C'_2}{\sqrt{(r - q_2)(q_1 - r + 1)}}$$

in normalized form. As before, the upper index on  $\mathbb{L}_r^k$

refers to the representation label of the subgroup; the lower index designates the unitary group order immediately before that subgroup.

In conclusion, the full base of the lowest - energy  $U_3$  irreducible representation  $(k, k_2)$  within a given Young pattern  $[f]$  can be generated by operators  $L'_3, L_3$  and  $L'_2$  of (287) in the chain  $U_3 \supset U_2 \supset U_1$ . The desired  $[f]$  symmetry function of maximum weight is constructed by prescription (79). The  $(k, k_2)$  representation corresponding to it is given by (269) which is the one lying lowest in energy and of maximum weight in  $U_3$ . One then applies (287) to build the full base but with the  $U_3$  lowering generators  $C_{\sigma}^{\sigma'} (\sigma > \sigma')$  replaced by linear combinations (138) of  $U_r$  lowering generators  $b_{\mu}^{\mu'} (\mu > \mu')$  which acting on a  $b_{\mu s}^+$  convert it into a  $b_{\mu s}^+$ . Thus, making that replacement one has the normalized lowering operators

$$L'_3 = \frac{1}{\sqrt{(q_1 - q_2 + 1)(k_1 - q_1 + 1)(q_1 - k_2)(q_1 + 1)}} \times$$

$$\times [(q_1 - q_2 + 1)(\sqrt{2} b'_3 + \sqrt{2} b_6^2 + b_5^2) + (\sqrt{2} b'_2 + \sqrt{2} b_4^2 + b_5^3)(b_3^2 + \sqrt{2} b_5^4 + \sqrt{2} b_6^5)]$$

$$L_3^2 = \sqrt{\frac{(q_1 - q_2 + 2)}{(k_2 - q_2 + 1)(k_1 - q_2 + 2)q_2}} (b_3^2 + \sqrt{2} b_5^4 + \sqrt{2} b_6^5) \quad (289a, b, c)$$

$$L'_2 = \frac{1}{\sqrt{(r_1 - q_2)(q_1 - r_1 + 1)}} (\sqrt{2} b'_2 + \sqrt{2} b_4^2 + b_5^3)$$

which by definition accomplish the following:

$$\mathbb{L}_3' P_{q_1 q_2, q_1}^{(k, k_2)} \equiv P_{q_1-1 q_2, q_1-1}^{(k, k_2)}$$

$$\mathbb{L}_3^2 P_{q_1 q_2, q_1}^{(k, k_2)} \equiv P_{q_1 q_2-1, q_1}^{(k, k_2)} \quad (290 a, b, c)$$

$$\mathbb{L}_2' P_{q_1 q_2, r_1}^{(k, k_2)} \equiv P_{q_1 q_2, r_1-1}^{(k, k_2)}$$

the row  $(q_1, q_2, r_1)$  of  $(k, k_2)$  from (92) being subject to the conditions

$$\begin{aligned} SU_3 \supset U_2 \supset U_1: \quad & k_1 \geq q_1 \geq k_2 \geq q_2 \geq 0 \\ & q_1 \geq r_1 \geq q_2 \end{aligned} \quad (291)$$

analogous to  $-l \leq m \leq l$  for  $R_3 \supset R_2$ . Just as the latter gives a dimensionality of  $(2l+1)$  for the  $R_3$  irreducible representation  $l$ , one can deduce<sup>45)</sup> from (291) the dimensionality of a given  $(k, k_2)$  of  $SU_3$ ,

$$D(SU_3) = \frac{1}{2}(k_1 - k_2 + 1)(k_1 + 2)(k_2 + 1) \quad (292)$$

this being a special case of the Weyl formula<sup>70)</sup> for  $U_r$ .

## 2. Alternative: $SU_3$ Wigner Coupling Coefficients.

We recall from (237) that classification by the cano-

nical chain  $U_3 \supset U_2 \supset U_1$  with a basis set

$$P_{q_1, q_2, r_1}^{(k_1, k_2)} (b_{\mu s}^+) |0\rangle \equiv \left| \begin{matrix} k_1, k_2, 0 \\ q_1, q_2 \\ r_1 \end{matrix} \right\rangle$$

was equivalent to classifying by  $U_3 \supset U_2 \supset R_2$  with basis set

$P_{q_1, q_2, M_L}^{(k_1, k_2)}$  where  $M_L = 2r_1 - q_1 - q_2$  is the projection of the total orbital angular momentum along a given  $z$ -axis. Taking the generators  $C_1', C_2', C_1^2$  and  $C_2^2$  of  $U_2$  in the three linear combinations

$$\begin{aligned} T_{+1} &\equiv \frac{1}{\sqrt{2}} C_1^2 & T_{-1} &\equiv -\frac{1}{\sqrt{2}} C_2^2 \\ T_0 &\equiv \frac{1}{2} (C_1' - C_2^2) \equiv \frac{1}{2} \mathcal{L}_0 \end{aligned} \quad (293)$$

one can directly verify them to be the normalized generators of  $SU_2$  (homomorphic to  $R_3$ ) since, upon using relations (135),

$$[T_0, T_{\pm 1}] = \pm T_{\pm 1} ; [T_{\pm 1}, T_{\mp 1}] = \mp T_0$$

is a Lie algebra identical to (142b) of  $R_3$ . Thus, the irreducible basis of  $SU_2 \supset R_2$  can be characterized by the eigenvalues of  $T^2$  and  $T_0$ , say  $t(t+1)$  and  $\tau$ , or by the kets  $|t\tau\rangle$  in analogy to the  $|lm\rangle$  of  $R_3 \supset R_2$ .

One can immediately find that

$$t = \frac{1}{2} (q_1 - q_2)$$

$$\tau = r - \frac{1}{2}(q_1 + q_2) = \frac{M_1}{2} \quad (294)$$

$$\tau = t, t-1, \dots, -t$$

so that our base can be designated by  $P_{q_1, q_2, \tau}^{(k_1, k_2)}$ .

The outer product of two irreducible representations  $D^{(l_1)}$  and  $D^{(l_2)}$  of  $R_3$  gives the well known Clebsch - Gordan equation

$$D^{(l_1)} \otimes D^{(l_2)} = D^{(l_1+l_2)} + D^{(l_1+l_2-1)} + \dots + D^{(l_1-l_2)}$$

where  $+$  means that the resulting irreducible representations are of the form

$$D^{(l_1)} \otimes D^{(l_2)} = \begin{pmatrix} \boxed{D^{(l_1+l_2)}} & & & 0 \\ & \boxed{D^{(l_1+l_2-1)}} & & \\ 0 & & \dots & \\ & & & \boxed{D^{(l_1-l_2)}} \end{pmatrix}$$

The basis of a given irreducible representation  $l = l_1 + l_2$ ,  $l_1 + l_2 - 1, \dots, l_1 - l_2$  is then simply

$$|lm\rangle = \sum_{m_1, m_2} \langle l, l_2, m_1, m_2 | lm \rangle |l_1, m_1\rangle |l_2, m_2\rangle. \quad (295)$$

For unitary groups, the outer product of two irreducible representations labelled by  $[f']$  and  $[f'']$  is

$$D^{[f']} \otimes D^{[f'']} = D^{[f_1]} + D^{[f_2]} + \dots$$



where the resulting  $[f] = [f_1], [f_2], \dots$  are determined by Littlewood's rules (See Appendix A) and the basis transforming irreducibly according to a given resulting  $[f]$  is in principle obtainable by generalized (Clebsch - Gordan, or ) Wigner coefficients for the unitary group in question.

In particular, for  $SU_3$  one has  $[f'] = (k'_1 k'_2)$  and  $[f''] = (k''_1 k''_2)$  whose rows are respectively given by  $q'_1 q'_2 \tau'$  and  $q''_1 q''_2 \tau''$  so that the full base corresponding to a given  $(k_1 k_2)$  resulting from  $(k'_1 k'_2) \otimes (k''_1 k''_2)$  shall be <sup>66)</sup>

$$P_{q_1 q_2 \tau}^{(k_1 k_2)} = \sum_{\substack{q'_1 q'_2 \tau' \\ q''_1 q''_2 \tau''}} \langle (k'_1 k'_2) q'_1 q'_2 \tau'; (k''_1 k''_2) q''_1 q''_2 \tau'' | (k_1 k_2) q_1 q_2 \tau \rangle P_{q'_1 q'_2 \tau'}^{(k'_1 k'_2)} P_{q''_1 q''_2 \tau''}^{(k''_1 k''_2)} \quad (296)$$

in analogy <sup>66)</sup> to (295) for  $R_3$ . The Wigner  $SU_3$  coupling coefficient factorizes into parts referring to  $SU_3 \supset SU_2$  and  $SU_2 \supset R_2$ :

$$\langle (k'_1 k'_2) q'_1 q'_2 \tau'; (k''_1 k''_2) q''_1 q''_2 \tau'' | (k_1 k_2) q_1 q_2 \tau \rangle = \quad (297)$$

$$\langle (k'_1 k'_2) q'_1 q'_2; (k''_1 k''_2) q''_1 q''_2 | (k_1 k_2) q_1 q_2 \rangle \langle t' \tau' t'' \tau'' | t \tau \rangle$$

$$t' = \frac{1}{2}(q'_1 - q'_2), \quad t'' = \frac{1}{2}(q''_1 - q''_2), \quad t = \frac{1}{2}(q_1 - q_2)$$

the latter factor being the ordinary Clebsch Gordan coefficient widely tabulated. The first factor in (297), called the reduced Wigner  $SU_3$  coefficient, was obtained by Moshinsky in closed algebraic form <sup>66)</sup> and extensive tables for them are being prepared by T. A. Brody <sup>71)</sup> at the University of Mexico.

To construct a given N - particle basis transforming irreducibly under  $U_3 \supset U_2 \supset U_1$  one could use (296) and (297)

to couple two particles, then a third, and so on up to  $N$  particles. The one - particle Young partition  $[1]$  of  $\mathcal{U}_r$  contains one and only one  $SU_3$  irreducible representation  $(k, k') = (r, 0)$ , where  $r = \frac{1}{2}(v+1)(v+2)$ . (The dimensionality of the  $SU_3$  representation is, by (292), simply  $r$ .) (The dimensionality of  $[1]$  of  $\mathcal{U}_r$  is, by the Weyl formula for  $\mathcal{U}_r$ , also  $r$ .) For the  $2s-1d$  shell  $r=6$  states, those given in (137) single-particle quantum numbers  $\mu \rightarrow n_1 n_2 n_0$ . Conditions (291) tell us that, since

$$2 \geq q_1 \geq 0 \geq q_2 \geq 0$$

$$q_1 \geq r_1 \geq q_2$$

then  $q_2 \equiv 0$  and

$$q_1 = \begin{matrix} 2 & 1 & 0 \end{matrix}$$

$$r_1 = \begin{matrix} 2 & 1 & 0 \end{matrix}$$

$$\begin{matrix} 1 & 0 \end{matrix}$$

$$0$$

will define the 6 states  $\mu \rightarrow q_1 q_2 r_1$  or  $\mu \rightarrow q_1 q_2 \tau$  recalling that  $\tau = r_1 - \frac{1}{2}(q_1 + q_2)$ . The correspondence between set  $(n_1, n_2, n_0)$  and set  $(q_1, q_2, \tau)$  is one - to - one and is given in Table VI.2.1.

TABLE VI.2.1

$\mu$	1	2	3	4	5	6
$n_1 n_2 n_0$	200	110	101	020	011	002
$q_1 q_2 \tau$	201	200	10 $\frac{1}{2}$	20 $\bar{1}$	10 $-\frac{1}{2}$	000
$P_{q_1 q_2 \tau}^{(20)} =$	$\Delta'_1$	$\Delta'_2$	$\Delta'_3$	$\Delta'_4$	$\Delta'_5$	$\Delta'_6$

In the last row of our table are given the single-particle functions: Using prescriptions (79) for [1] of  $U_6$  one gets  $P = b_{11}^+$ , which by (138) and (144) is of maximum weight in  $U_3$  so that

$$P \equiv P_{201}^{(20)}(b_{\mu s}^+) \equiv b_{11}^+ \equiv \Delta_1^+ (\mu=1, s=1) \quad (298)$$

the remaining states labeled from 2 to 6 in order of decreasing weight  $(\eta, \eta_1, \eta_0)$  will thus correspond respectively to  $(\mu s) = (2,1)(3,1)(4,1)(5,1)(6,1)$ . (These could also be obtained by successive application of lowering operators (289) on maximum weight state (298).) As in Appendix A, we shall use the simplified notation

$$\Delta_{\mu_1 \mu_2 \dots \mu_r}^{s_1 s_2 \dots s_r} \equiv \sum_{P_s} (-1)^P P_s b_{\mu_1 s_1}^+ b_{\mu_2 s_2}^+ \dots b_{\mu_r s_r}^+ \quad (299)$$

where  $P_s$  is a permutation of the indices  $s_1 s_2 \dots s_r$  so that  $\Delta_{\mu_1 \mu_2 \dots \mu_r}^{s_1 s_2 \dots s_r}$  is antisymmetric with regards to spin - isos in indices  $s$  and, since the  $b_{\mu s}^+$  anti-commute, symmetric with respect to permutation of orbital indices  $\mu$ . In short, a form  $\Delta_{\mu_1 \mu_2 \mu_3}^{s_1 s_2 s_3}$ , say, transforms under  $U_r$  like [f] = [3] and under  $U_2$  like [f] = [11]. Thus of course,  $\Delta_{\mu}^s \equiv b_{\mu s}^+$ .

To couple two  $2s-1d$  shell particles one has, by Littlewood's rules, in  $SU_3$ :

$$(20) \otimes (20) = (40) + (22) + (31)$$

of which (40) and (22) are contained in  $[2]$  and  $(31) \subset [11]$  (see Table V.1.1). Representation (40) is the lowest - energy

$SU_3$  representation of both  $O^8$  and  $F^8$ . Should we then construct the base in  $(k, k_2) = (40)$  by the Wigner coefficient method (296), our states  $P_{q, q_2, \tau}^{(40)}$  would be linear combinations of  $\Delta_{\mu'}^1 \Delta_{\mu''}^2$  which terms would have to be regrouped to give terms of the type  $\Delta_{\mu'}^{12}$ , that is, of symmetry  $[f] = [2]$  under  $U_6$ . If the permutation symmetry is maintained throughout the whole construction however, great labor will be saved. Thus, to construct  $P_{q, q_2, \tau}^{(40)}$  it is preferable to use the lowering operator method beginning with

$$P_{40, 2}^{(40)} = b_{11}^+ b_{12}^+ \equiv \frac{1}{2} \Delta_{11}^{12},$$

which is of maximum weight in  $U_3 \supset U_2 \supset U_1$ . The symmetry  $[f] = [2]$  shall then be maintained throughout as operations (289) affect only the lower indices of  $\Delta_{11}^{12}$  without altering the form  $\Delta_{\mu'}^{12}$ .

Efficient construction of an  $N$  - particle  $U_3 \supset U_2 \supset U_1$  base for a definite  $(k, k_2)$  contained in a given  $[f]$  partition requires an adequate combination of both methods discussed above. By expressing  $N$  - particle states in terms of determinants of the type (299) one avoids destroying the permutation symmetry of the desired partition  $[f]$ . Consider the  $N = 3$  particle partition  $[f] = [431]$ .

According to formula (269), the lowest - energy  $SU_3$  representation contained here is  $(k, k_2) = (11, 2)$ . To construct our 8 particle base  $P_{q,q,\tau}^{(11,2)}$  we proceed as follows:

- 1) Construct by prescription (79) the maximum weight polynomial for  $(k, k_2) = (80)$ , this being the lowest-energy  $SU_3$  representation contained in  $[f] = [4]$ . Applying lowering operators (289), proceed to generate the full base  $P_{q,q,\tau}^{(80)}$ . This is simple since here  $k_2 = 0$ . Each of the  $D(U_3) = 36$  polynomials (see formula (292)) of the full base will be simple linear combinations of terms  $\Delta_{\mu_1 \mu_2 \mu_3 \mu_4}^{1234}$  with symmetry  $[f] = [4]$ .
- 2) Repeat the process for  $(k, k_2) = (60)$  which is the lowest - energy  $SU_3$  representation contained in  $[f] = [3]$ , i.e., construct the  $D(U_3) = 28$  polynomials  $P_{q,q,\tau}^{(60)}$ .
- 3) The lowest  $(k, k_2)$  of  $[f] = [43]$  is by (269) equal to  $(11, 3)$  so that we then couple  $P_{q',q',\tau'}^{(80)}$  and  $P_{q'',q'',\tau''}^{(60)}$  obtained in (1) and (2) by (296) to give us  $P_{q,q,\tau}^{(11,3)}$ . These will result as linear combinations of  $\Delta_{\mu_1 \mu_2 \mu_3 \mu_4}^{1234}$  and  $\Delta_{\mu_1' \mu_2' \mu_3'}^{123}$  with obvious symmetry  $[f] = [43]$ .
- 4) Finally, couple  $P_{q,q,\tau}^{(11,3)}$  with  $P_{q''',q''',\tau'''}^{(20)}$  of Table to give  $P_{q,q,\tau}^{(11,2)}$  by method (296). These are linear combinations of  $\Delta_{\mu_1' \mu_2' \mu_3' \mu_4'}^{1234}$  and  $\Delta_{\mu_1'' \mu_2'' \mu_3''}^{123}$  and of symmetry  $[f] = [431]$  as desired.

Notice that permutational symmetry has been maintained in each step. There will be  $D(U_3) = 195$  polynomials for  $(k, k_2) = (11, 2)$  but one need only construct those  $P_{q,q,\tau}^{(11,2)}$  with  $\tau$  non-negative as negative values of  $M_L = 2\tau$  are unnecessary to calculate matrix elements of a central interaction like pairing.

### 3. Explicit Construction of the $F^{20}$ Base for Lowest $(k, k_2)$ in $SU_3 \supset U_2 \supset R_2$ .

In  $F^{20}$  one has 4 nucleons (1 proton and 3 neutrons) in the  $2s-1d$  shell. Lowest levels are thus given by  $T=1$ . The most symmetric Young partition of  $N=4$  with conjugate representation containing  $T=1$  is, according to Appendix A,  $[f] = [31]$ . The lowest energy  $(k, k_2)$  value contained in  $[31]$  from Table X.1.1 is (71). Contributions to the wave function proceeding from higher-energy  $(k, k_2)$  values of will be neglected (Elliott hypothesis) in our calculations.

The polynomial basis  $P_{q,q_2,\tau}^{(71)}(b_{\mu s}^+)$  is composed of  $D(U_3:(71)) = 63$  polynomial components of which only 35 possess  $\tau$  non-negative. This set of 35 polynomials was constructed separately by both lowering - operator and the combined technique explained in the last section. Identical results were obtained but the combined technique involved much less labor and time. Since the chain of representations  $[31] \supset (71)$  is desired, one starts by using lowering operators (289) on the maximum weight  $(k, k_2) = (60)$  function of symmetry  $[3]$  :

$$P_{60,3}^{(60)} = b_{11}^+ b_{12}^+ b_{13}^+ \equiv \frac{1}{6} \Delta_{111}^{123}$$

in accordance with (79) and definition (299). A set of  $D(U_3:(60)) = 28$  normalized polynomials is generated, each of which is a linear combination of terms  $\Delta_{\mu_1 \mu_2 \mu_3}^{123}$ . The longest polynomial of this set was, for example,

$$P_{40,0}^{(60)} = \frac{1}{3\sqrt{10}} \left( \Delta_{622}^{123} + 2\sqrt{2} \Delta_{532}^{123} + \Delta_{334}^{123} + \Delta_{641}^{123} + \Delta_{551}^{123} \right).$$

(Incidentally, this  $(k, k_2) = (60)$  base corresponds to the lowest energy  $SU_3$  representation of  $F^{19}$ ).

Next, we add the 4th particle to our 3 - particle  
 $P_{q_1, q_2, \tau}^{(60)}$  using  $SU_3$  Wigner coefficients. The fourth particle  
 is  $(k_1, k_2) = (20)$  so that

$$(60) \otimes (20) = (80) + (71) + (62)$$

where the first, second and third resulting  $(k_1, k_2)$  values correspond to the lowest energy  $SU_3$  representations of  $Ne^{20}$ ,  $F^{20}$  and  $O^{20}$ , respectively (see Table V.1.1). We take the (71), and thus require the  $SU_3$  reduced Wigner coefficients

$$\langle (60) q_1' 0; (20) q_1'' 0 \rangle (71) q_1 q_2 \rangle$$

$$6 \geq q_1' \geq 0 \quad 2 \geq q_1'' \geq 0 \quad 7 \geq q_1 \geq 1 \geq q_2$$

which are available in Brody's tables.<sup>71)</sup> One also requires the ordinary Clebsch - Gordan coefficients

$$\langle t' \tau' t'' \tau'' | t \tau \rangle = \langle \frac{1}{2} q_1' \tau' \frac{1}{2} q_1'' \tau'' | \frac{1}{2} (q_1 - q_2) \tau \rangle$$

$$\text{for } \tau = \frac{1}{2} (q_1 - q_2), \frac{1}{2} (q_1 - q_2) - 1, \dots, \begin{cases} 0 \\ \frac{1}{2} \end{cases}$$

which are also tabulated in various sources. Then by (296) and (297) the  $F^{20}$  lowest - energy  $U_3 \supset U_2 \supset R_2$  base is gotten by evaluating, for  $\tau \geq 0$ ,

$$P_{q_1, q_2, \tau}^{(71)} (\Delta_{\mu_1}^1, \Delta_{\mu_2, \mu_3, \mu_4}^{1, 2, 3}) =$$

$$\sum_{q_1' q_1''} \langle (60) q_1' 0; (20) q_1'' 0 \rangle (71) q_1 q_2 \rangle \sum_{\tau' \tau''} \langle \frac{1}{2} q_1' \tau' \frac{1}{2} q_1'' \tau'' | \frac{1}{2} (q_1 - q_2) \tau \rangle \times$$

$$\times P_{q_1' 0 \tau'}^{(60)} (\Delta_{\mu_2 \mu_3 \mu_4}^{123}) P_{q_1'' 0 \tau''}^{(20)} (\Delta_{\mu_1}') \quad (300)$$

where single-particle functions  $P_{q_1' 0 \tau'}^{(20)}$  are those of Table VII.2.1. The 35 polynomials  $P_{q_1' 0 \tau'}^{(71)}$  of  $\tau \geq 0$  are linear combinations of terms  $\Delta_{\mu_1 \mu_2 \mu_3}^{123} \Delta_{\mu_4}'$  (which have symmetry [31]). Let us abbreviate

$$\Delta_{\mu_1}' \Delta_{\mu_2 \mu_3 \mu_4}^{123} \equiv (\mu_1, \mu_2 \mu_3 \mu_4) \quad (301)$$

as the upper (spin - isospin) indices are identical to every term. The polynomial  $P_{71,3}^{(71)}$  resulting from (300) is however

$$P_{71,3}^{(71)} = \frac{\sqrt{3}}{12} [(2, III) - (1, 2 II)] = \frac{2}{\sqrt{3}} [\frac{1}{6} (2, III)] \quad (302)$$

since (as is easily shown)  $\Delta_1' \Delta_{211}^{123} = -\frac{1}{3} \Delta_2' \Delta_{111}^{123}$ . Prescription (79) allows us, on the other hand, to construct this polynomial immediately as it is of maximum weight in  $SU_3$  namely:

$$P_{71,3}^{(71)} = b_{11}^+ b_{12}^+ b_{13}^+ b_{21}^+ \equiv \frac{1}{6} (2, III)$$

which is moreover normalized. Thus, in view of (302), we must multiply by  $\sqrt{3}/2$  all polynomials resulting from (300) in order to obtain a normalized set. Moreover, the number of terms  $\Delta_{\mu_1}' \Delta_{\mu_2 \mu_3 \mu_4}^{123}$  resulting in each  $P_{q_1' 0 \tau'}^{(71)}$  from (300) can be reduced by use of the following easily derived identities:



$$(a, bbb) = -3(b, abb)$$

$$(a, bbc) + 2(b, abc) + (c, bba) = 0 \quad (303)$$

$$(a, abb) = -(b, aba)$$

$$(a, bcd) + (b, acd) + (c, abd) + (d, abc) = 0.$$

The complete list of  $P_{q,q,\tau}^{(71)}$  for  $\tau \geq 0$  is given in table where  $\lambda \equiv (q, q, \tau) = 1, 2, \dots, 35$ .

TABLE VI.3.1

$q, q, \tau$ ②	$[51](71)q, q, \tau; [21]S=1, M_S=S, T=1, M_T=T \rangle \equiv P_{q,q,\tau}^{(71)}  0\rangle$
713 ①	$\frac{1}{6}(2, 111)$
603 ②	$\frac{1}{6\sqrt{3}}[3(3, 311) + (6, 111)]$
712 ③	$\frac{1}{6\sqrt{3}}[3(2, 211) + (4, 111)]$
512 ④	$\frac{1}{24\sqrt{5}}[12(2, 331) + 7\sqrt{2}(5, 311) + 4(3, 321) + 6(3, 611) + 2(6, 211) + \sqrt{2}(3, 511)]$
602 ⑤	$\frac{1}{6\sqrt{2}}[2\sqrt{2}(3, 321) + \sqrt{2}(6, 211) + (5, 311) + (3, 511)]$
402 ⑥	$\frac{1}{2\sqrt{10}}[2(3, 631) + 2(6, 331) + (6, 611)]$
711 ⑦	$\frac{1}{2\sqrt{15}}[2(2, 221) + 2(4, 211) + (2, 411)]$

511	⑧	$\frac{1}{24\sqrt{5}} \left[ 4\sqrt{2}(2,332) + 16(5,321) + 6\sqrt{2}(2,621) + 2\sqrt{2}(6,221) + 4(3,521) \right. \\ \left. + 6\sqrt{2}(4,331) + 2\sqrt{2}(3,431) + 3\sqrt{2}(4,611) + \sqrt{2}(6,411) + 12(2,531) \right. \\ \left. + 4\sqrt{2}(5,511) \right]$
311	⑨	$\frac{1}{2\sqrt{30}} \left[ (2,661) - (6,332) - 3(3,632) + (6,621) \right] \\ + \frac{1}{12\sqrt{5}} \left[ 15(5,631) + 6(6,531) + 3(6,51) + 4(5,333) \right]$
601	⑩	$\frac{1}{6\sqrt{5}} \left[ 2(3,322) + 2(6,221) + 2\sqrt{2}(5,321) + 2\sqrt{2}(3,521) \right. \\ \left. + 2(3,341) + (6,411) + (5,511) \right]$
401	⑪	$\frac{1}{2\sqrt{10}} \left[ \sqrt{2}(3,632) + \sqrt{2}(6,332) + \sqrt{2}(6,621) + (5,631) + (3,651) + 2(6,531) \right]$
201	⑫	$\frac{1}{2\sqrt{3}} \left[ (6,633) + (6,661) \right]$
710	⑬	$\frac{1}{2\sqrt{10}} \left[ 2(4,221) + 2(2,421) + (4,411) \right]$
510	⑭	$\frac{1}{4\sqrt{30}} \left[ \sqrt{2}(5,322) - \sqrt{2}(3,522) + 2(4,332) - 2(3,432) + 2(4,621) \right. \\ \left. + 6(5,521) + 3\sqrt{2}(5,431) + \sqrt{2}(3,541) + 4\sqrt{2}(4,531) + 2(2,551) - 2(1,642) \right]$
310	⑮	$\frac{1}{2\sqrt{30}} \left[ (4,661) - (6,433) - 3(3,643) + (6,641) + \sqrt{2}(5,632) - \sqrt{2}(3,652) \right. \\ \left. + 3(5,651) + (6,551) + 2(5,533) \right]$
110	⑯	$\frac{1}{4\sqrt{2}} \left[ (1,663) - (3,665) \right]$
600	⑰	$\frac{1}{6\sqrt{5}} \left[ \sqrt{2}(1,222) + 3(5,322) + 3(3,522) + 3\sqrt{2}(3,432) + 3\sqrt{2}(6,421) \right. \\ \left. + 3\sqrt{2}(5,521) + 3(5,431) + 3(3,541) \right]$

400	(18)	$\frac{1}{2\sqrt{15}} [(6,622) + \sqrt{2}(5,632) + \sqrt{2}(3,652) + 2\sqrt{2}(6,532) + (3,643) + (6,433) + (6,641) + (5,651) + (6,551)]$
200	(19)	$\frac{1}{2\sqrt{6}} [2(6,653) + \sqrt{2}(6,662)]$
$61\frac{5}{2}$	(20)	$\frac{1}{24\sqrt{21}} [21\sqrt{2}(2,311) + 8(5,111) + 3\sqrt{2}(3,211)]$
$70\frac{5}{2}$	(21)	$\frac{1}{6\sqrt{7}} [3\sqrt{2}(3,211) + (5,111)]$
$50\frac{5}{2}$	(22)	$\frac{1}{2\sqrt{15}} [2(3,331) + (3,611) + 2(6,311)]$
$61\frac{3}{2}$	(23)	$\frac{1}{8\sqrt{105}} [28(2,321) + 9\sqrt{2}(5,211) + 4(3,221) + 14(4,311) + 2(3,411) + 7\sqrt{2}(2,511)]$
$41\frac{3}{2}$	(24)	$\frac{1}{60} [4(2,333) + 15(2,631) + 6(6,321) + 3(3,621)] + \frac{1}{20\sqrt{2}} [6(5,331) + 3(5,611) + 2(3,531) + (6,511)]$
$70\frac{3}{2}$	(25)	$\frac{1}{2\sqrt{21}} [2(3,221) + \sqrt{2}(5,211) + (3,411)]$
$50\frac{3}{2}$	(26)	$\frac{1}{10\sqrt{3}} [2\sqrt{2}(3,332) + 2\sqrt{2}(3,621) + 4\sqrt{2}(6,321) + 2(5,331) + 4(3,531) + 2(6,511) + (5,611)]$
$30\frac{3}{2}$	(27)	$\frac{1}{6\sqrt{15}} [12(6,631) + 2(6,333) + 3(3,661)]$
$61\frac{1}{2}$	(28)	$\frac{1}{12\sqrt{210}} [30(5,221) - 4\sqrt{2}(3,222) + 42\sqrt{2}(4,321) + 9\sqrt{2}(3,421) + 42(2,521) + 21\sqrt{2}(2,431) + 15(5,411) + 21(4,511)]$
$41\frac{1}{2}$	(29)	$\frac{1}{60\sqrt{3}} [3\sqrt{2}(2,632) - 3\sqrt{2}(3,622) + 4\sqrt{2}(4,333) + 9\sqrt{2}(4,631) - 6\sqrt{2}(1,643) - 3\sqrt{2}(3,641) - 18(3,532) + 3(2,651) - 12(1,652) + 9(5,621) + 6(5,332) + 21\sqrt{2}(5,531) + 3\sqrt{2}(3,551)]$

30	$\frac{1}{6\sqrt{2}} [-(6,632) - (3,662)] + \frac{1}{12} [3(5,633) + 2(5,661) + (6,533) + 2(6,651)]$
31	$\frac{1}{6\sqrt{35}} [2\sqrt{2}(3,222) + 6(5,221) + 6\sqrt{2}(3,421) + 3(5,411)]$
32	$\frac{1}{5\sqrt{6}} [(3,622) + 2(6,322) - \sqrt{2}(3,533) + 2\sqrt{2}(6,521) + \sqrt{2}(5,621) + (3,433) + (3,641) + 2(6,431) - (1,553)]$
33	$\frac{1}{6\sqrt{5}} [4\sqrt{2}(6,632) + \sqrt{2}(3,662) + 4(6,651) + 2(6,533) + (5,661)]$
34	$\frac{1}{2} [(6,663)]$
35	$\frac{1}{6} [(3,111)]$

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The base  $P_{q,q,\tau}^{(71)}$  given in terms of  $(\mu_1, \mu_2, \mu_3, \mu_4)$  is convenient as any interaction operator is expressible, from (35) and (36), in terms of operators  $b_{\mu_i}^{\mu_j}$  which replace  $\mu_2$  by  $\mu_1$ , that is

$$\begin{aligned} b_{\mu_1}^{\mu_2}(\mu_1, \mu_2, \mu_3, \mu_4) &\equiv (\mu_1, \mu_1, \mu_3, \mu_4) \\ b_{\mu_1}^{\mu_5}(\mu_1, \mu_2, \mu_3, \mu_4) &\equiv 0, \text{ etc.} \end{aligned} \quad (304)$$

To construct the base for any number of particles N in 2s-1d shell with orbital symmetry  $[f]$  and associated lowest-energy  $SU_3$  representation  $(k_1, k_2)$ , one can proceed in the following systematic fashion. If one has the bases for  $N=1, 2, 3$  and 4 particles with  $[f](k_1, k_2) = [1](20), [2](40), [3](60)$  and  $[4](80)$ , respectively, any  $[f](k_1, k_2)$  for any N can then be built by appropriately combining these "elemental bases" with the aid of  $SU_3$  Wigner coupling coefficients in accord with equation (296). The  $[1](20)$  base ( $O^{17}$  and  $F^{17}$ ) is simply given in Table VI.2.1. Base  $[2](40)$  ( $O^{18}$  and  $F^{18}$ ) could easily be generated from  $\mathbb{P}^{(40)} = b_{11}^+ b_{12}^+$  (of orbital symmetry  $[2]$ ) by the lowering operator method (290). The  $[3](60)$  base ( $F^{19}$  and  $Ne^{19}$ ) can likewise be gotten from  $\mathbb{P}^{(60)} = b_{11}^+ b_{22}^+ b_{13}^+$ , of orbital symmetry  $[3]$  (See VI. 3). Finally, the base  $[4](80)$  ( $Ne^{20}$ ) follows  $\mathbb{P}^{(80)} = b_{11}^+ b_{12}^+ b_{13}^+ b_{14}^+$ , with orbital symmetry  $[4]$ . By formula (292) the respective dimensionalities are 6, 15, 28 and 40 but, as mentioned before, one may restrict oneself to components of the base having positive  $M_L$  row-index values. A few schematic examples follow:

$$\underline{(k'_1, k'_2) \otimes (k''_1, k''_2) = \dots + (k_1, k_2) + \dots \text{ (in formula 296)}}.$$

$$[3](60) \otimes [1](20) = \dots + [31](71) + \dots \quad (F^{20}, Na^{20})$$

$$[4](80) \otimes [3](71) = \dots + [431](11,2) + \dots \quad (Na^{24}, Al^{24})$$

$$[4](80) \otimes [43](11,2) = \dots + [4431](12,3) + \dots \quad (Al^{28})$$

$$[4](80) \otimes [4](80) = \dots + [44](12,4) + \dots \quad (Mg^{24})$$

$$[1](20) \otimes [44](12,4) = \dots + [441](12,3) + \dots \quad (Al^{25})$$

$$[2](40) \otimes [44](12,4) = \dots + [442](11,1) + \dots \quad (Na^{26})$$

$$[1](20) \otimes [442](11,1) = \dots + [4421](12,4) + \dots \quad (Al^{27}, Si^{27})$$

$\dots$  etc.  $\dots$  etc.  $\dots$  etc.  $\dots$

Global studies using the  $SU_3$  classification scheme can thus be carried out for several nuclei at a time, leaving as much as possible to the computer.

CHAPTER

VII. PAIRING FORCE MATRICES.

The many - orbital pairing force matrices in the physical (angular momentum) scheme  $\mathcal{U}_r \supset \mathcal{U}_3 \supset R_3 \supset R_2$  for a single  $SU_3$  representation  $(k, k_2)$  for N particles in an oscillator shell are designated by

$$\| \langle [f] \alpha(k, k_2) \omega' L M_L | P | [f] \alpha(k, k_2) \omega L M_L \rangle \| \quad (305)$$

where  $P$  is the pairing force operator (219). Since  $P$  was assumed a central spin - isospin - independent operator, then  $[L^2, P] = [L_0, P] = 0$  so that (Appendix F) it cannot connect different  $L$ -values in (305) and is moreover independent of the row label  $M_L$ .

In the canonical scheme  $\mathcal{U}_r \supset \mathcal{U}_3 \supset \mathcal{U}_2 \supset \mathcal{U}_1$  our matrices would be denoted by

$$\| \langle [f] \alpha(k, k_2) q_1' q_2' M_L | P | [f] \alpha(k, k_2) q_1 q_2 M_L \rangle \| \quad (306)$$

whose elements can be evaluated readily if states  $| [f] \alpha(k, k_2) q_1 q_2 M_L \rangle$  are constructed as shown in the previous Chapter since the effect of operator (219) on these states, as well as the resulting scalar products, is straightforward. This will be illustrated shortly.

In view of transformation (I.1) of Appendix I, the passage from matrices (306) to (305) would then involve the similarity transformation

$$\| P \|_{R_1} = \| S \| \times \| P \|_{\mathcal{U}_2} \times \| \tilde{S} \|, \quad (307)$$



the  $R_3$  and  $U_2$  subscripts referring respectively to (305) and (306), and  $\|S\|$  standing for orthogonal transformation matrices of the type (I.9) of Appendix I.

If, however,  $\omega' = \omega$  in (305) --- which is the case for the  $(k, k_2) = (71)$  of our present interest --- the matrix is diagonal and of dimension equal to as many  $L$  - values as are contained in  $(k, k_2)$ . Hence, in view of (307), the sum of the eigenvalues of  $P$ , which are labeled by  $L$ , is simply

$$\begin{aligned} \text{Tr} \| \langle [f] \alpha(k, k_2) \omega L M_L | P | [f] \alpha(k, k_2) \omega L M_L \rangle \| = \\ \text{Tr} \| \langle [f] \alpha(k, k_2) q_1 q_2' M_L | P | [f] \alpha(k, k_2) q_1 q_2 M_L \rangle \| \end{aligned} \quad (308)$$

so that only the diagonal elements of (306) need be calculated to obtain the eigenvalues of (305) that will figure in the calculation of energy levels. Matrix (306) is explicitly reduced into submatrices labeled by  $M_L = L_{\max}, L_{\max} - 1, \dots, 1, 0$  (where  $L_{\max} = k_1$ ) and of dimensions equal to the multiplicity of each  $M_L$  involved. Matrix (305), which is diagonal, can be imagined to be ordered into groups of diagonal elements labeled also by the  $M_L$ 's. Then, according to (308), the eigenvalues  $p_L$  of  $P$  associated with a definite  $L$  can readily be found through

$$\begin{aligned} p_L = \left\{ \text{Tr} \| \langle [f] \alpha(k, k_2) q_1 q_2' M_L = L | P | [f] \alpha(k, k_2) q_1 q_2 M_L = L \rangle \| \right. \\ \left. - \text{Tr} \| \langle [f] \alpha(k, k_2) q_1 q_2' M_L = L+1 | P | [f] \alpha(k, k_2) q_1 q_2 M_L = L+1 \rangle \| \right\} \end{aligned} \quad (309)$$

We now turn to the evaluation of the diagonal elements of (306). Applying the numbering convention of Table VI.2.1 for  $\mu = (\eta, \eta_1, \eta_0)$  to pairing operator (219), which is bilinear in

$U_r$  group generators  $b_\mu^{\mu'}$  defined in (37), one obtains

$$\begin{aligned} P = & b_1^2 b_4^2 + b_1^6 b_4^6 + b_2^1 b_2^4 + b_6^1 b_6^4 + b_1^4 b_4^1 + b_3^5 b_5^3 \\ & + \frac{1}{2} b_2^6 b_2^6 + \frac{1}{2} b_6^2 b_6^2 + b_1^1 b_4^4 + b_3^3 b_5^5 - b_1^3 b_4^5 - b_1^5 b_4^3 \\ & - b_2^3 b_2^5 - b_3^1 b_5^4 - b_3^2 b_5^2 - b_3^4 b_5^1 - b_3^6 b_5^6 - b_6^3 b_6^5 \\ & - b_1^1 - b_3^3 + \frac{1}{2} b_2^2 (b_2^2 - 1) + \frac{1}{2} b_6^6 (b_6^6 - 1). \end{aligned} \quad (310)$$

Recalling (304), the effect of this operator  $P$  on states of the type given in Table VI.3.1 is clear; one must be careful to use the same numbering convention regarding  $\mu$  for both operator  $P$  and single - particle states  $P_{q,q_2\tau}^{(20)}$  as in Table VI.2.1. For  $(k_1, k_2) = (71)$ , operator (310) must be applied to all 35 polynomials given in Table VI.3.1 to get

$$P | [f] \lambda (k_1, k_2) q, q_2 M_L \rangle \equiv P P_{q, q_2 \tau = \frac{M_L}{2}}^{(71)} (\Delta_{\mu_1}^1, \Delta_{\mu_2 \mu_3 \mu_4}^{123}) | 0 \rangle \quad (311)$$

for all 35 cases. A typical result of (311) for the polynomial with  $q, q_2 \tau = 41 \frac{3}{2}$  is, having used simplifying relations (303) and regrouping terms,

$$P P_{41 \frac{3}{2}}^{(71)} = \frac{1}{10\sqrt{2}} \left[ 9(3,531) + 7(5,331) - 7(2,321) - (3,221) - 7(6,631) \right. \\ \left. - (3,661) + 5(3,411) + 7(1,431) \right] \quad (312)$$

where as in notation (301), e.g.,

$$(3,531) \equiv \Delta_3^1 \Delta_{531}^{123},$$

lower indices referring to orbital and upper to spin - isospin quantum numbers. Multiplication of (311) on the left by

$$\langle [f] \alpha(k, k_2) q_1 q_2 M_L | \equiv \langle 0 | (P_{q_1 q_2 \tau = M_{L_2}}^{(k, k_2)})^\dagger$$

will result in only a few different elemental scalar products

of the type  $(\Delta_{\mu_1}^1 \Delta_{\mu_2 \mu_3 \mu_4}^{123} \Delta_{\mu'_1}^1 \Delta_{\mu'_2 \mu'_3 \mu'_4}^{123}) \equiv (\mu_1 \mu_2 \mu_3 \mu_4 / \mu'_1 \mu'_2 \mu'_3 \mu'_4)$

which are simple to evaluate with the aid of anticommutation

relations (12) and definition (299). Calculation of diagonal

elements of (306) for  $(k, k_2) = (71)$  required only those

$(\mu_1 \mu_2 \mu_3 \mu_4 / \mu'_1 \mu'_2 \mu'_3 \mu'_4)$  given in the following list:

$$(a, bcd / a' b' c' d') \equiv \Delta_a^1 \Delta_{bcd}^{123}, \Delta_{a'}^1 \Delta_{b' c' d'}^{123} \quad (313)$$

$$\begin{array}{l|l|l} (a, bcd / a, bcd) = 6 & (a, bbb / b, abb) = -12 & (a, abb / a, abb) = 8 \\ (a, bbc / a, bbc) = 12 & (a, bcc / b, acc) = -4 & (a, abc / a, abc) = 4 \\ (a, bbb / a, bbb) = 36 & (a, bcc / c, abc) = -4 & (a, aab / a, aab) = 4 \\ & (a, bcd / b, acd) = -2 & \end{array}$$

If in  $(\mu_1, \mu_2, \mu_3, \mu_4 / \mu'_1, \mu'_2, \mu'_3, \mu'_4)$  the set of indices  $\mu_1, \mu_2, \mu_3, \mu_4$  differs from the set  $\mu'_1, \mu'_2, \mu'_3, \mu'_4$ , in any order, the elemental scalar product vanishes because of (11).

To finish our example for  $q_1 q_2 \tau = 413/2$  with  $(k, k_2) = (71)$  and  $[f] = [31]$ , ( $\Delta$  being unneeded):

$$\begin{aligned} \langle [31](71) 413 | P | [31](71) 413 \rangle &\equiv \langle 0 | (P_{413/2}^{(71)})^\dagger P P_{413/2}^{(71)} | 0 \rangle \\ &= \frac{1}{20 \cdot 10 \cdot 2} \left[ \underbrace{6 \cdot 9(5, 331 / 3, 531)}_{-4} + \underbrace{6 \cdot 7(5, 331 / 5, 331)}_{12} + \underbrace{2 \cdot 9(3, 531 / 3, 531)}_4 \right. \\ &\quad \left. + \underbrace{2 \cdot 7(3, 531 / 5, 331)}_{-4} \right] \end{aligned}$$

$$= \frac{1}{5 \cdot 10 \cdot 2} [-6.9 + 6.73 + 2.9 - 2.7] = \frac{76}{5 \cdot 10 \cdot 2} = \frac{19}{25}$$

Calculating the other 34 diagonal elements of (306) in a similar way and then using (309), one obtains the following eigenvalues  $p_L$  for  $L=1,2,\dots,7$  of (305):

Eigenvalues of  $\| \langle [31](7) L M_L | P | [31](7) L M_L \rangle \|$

L	1	2	3	4	5	6	7
$-p_L$	$-\frac{21}{5}$	$-\frac{23}{5}$	$-\frac{11}{5}$	$-\frac{11}{3}$	0	0	0

(314)

constituting five different levels the highest of which is degenerate in  $L = 5, 6, 7$ .

In cases of certain  $(k, k_2)$  values where  $\omega \neq \omega'$  in (305), the trace technique (309) would not suffice to obtain all the  $p_L$  eigenvalues as then matrix (305) is not entirely diagonal and would contain small submatrices along the diagonal for those  $L$ -values which occur multiply under the given  $(k, k_2)$ . These submatrices would be of dimension equal to the number of  $\omega$ 's that occur for the given  $L$  and to obtain the corresponding eigenvalues  $p_{\omega L}$  of  $\| P \|_{R_3}$  the similarity transformation (307) would be applied explicitly upon  $\| P \|_{U_2}$  for those  $M_L$ -labeled submatrices of  $\| P \|_{R_3}$  containing non-diagonal terms. The resulting  $\| P \|_{R_3}$  are then diagonalized and the  $p_{\omega L}$  obtained. A typical instance of this  $\omega \neq \omega'$  situation is found in Table for  $M_g^{24}$  with  $[f] = [44]$ ,  $(k, k_2) = (12, 4)$  which contains up to three - fold multiplicity in the  $L$ -structure, that is,  $\omega \neq \omega' \neq \omega''$ .

Result (314), after coupling each  $L$ -value with the associated total spin  $S$ -values of each level to give total  $J$ , will provide (diagonal) matrices with elements

$$\begin{aligned}
 & \langle [f](k_1 k_2) \omega' L S J M_J T | P | [f](k_1 k_2) \omega L S J M_J T \rangle = \\
 & \sum_{\substack{M_L' M_S' \\ M_L M_S}} \langle L S M_L' M_S' | J M_J \rangle \langle L S M_L M_S | J M_J \rangle \times \\
 & \times \langle [f](k_1 k_2) \omega' L M_L'; [\tilde{f}] S M_S' T | P | [f](k_1 k_2) \omega L M_L; [\tilde{f}] S M_S, T \rangle
 \end{aligned}$$

where the latter matrix element is identical with those of (305) as  $P$  is spin - isospin independent. This element moves outside the sum since it is independent of row labels  $M_L', M_S', M_L, M_S$  so that by Clebsch - Gordan Coefficient orthogonality

$$\begin{aligned}
 & \langle [f](k_1 k_2) \omega' L S J T | P | [f](k_1 k_2) \omega L S J T \rangle = \quad (315) \\
 & \langle [f](k_1 k_2) \omega' L; [\tilde{f}] S T | P | [f](k_1 k_2) \omega L; [\tilde{f}] S T \rangle
 \end{aligned}$$

being independent of row label  $M_J$ . Matrices (315) with rows labeled by  $L$  and  $S$  shall form part of the total interaction hamiltonian matrices whose eigenvalues are the energies of definite  $J$  and  $T$  values.

CHAPTER

VIII. SPIN-ORBIT FORCE MATRICES FOR  $F^{20}$ .

To permit evaluation of the spin-orbit interaction matrix elements (263) in the  $SU_3$  scheme one must evaluate coefficients  $A_{s's}$ ,  $B_{s's}$  and the simple matrix of elements  $\langle (k_1 k_2) \omega' L' \| \mathcal{K} \| (k_1 k_2) \omega L \rangle$ . This will be illustrated here for  $F^{20}$ ; extension to other nuclei will then be quite straightforward.

Coefficients  $A_{s's}$  and  $B_{s's}$  being independent of  $U_3$  row indices  $q, q', r$ , and of  $U_2$  (spin) row index  $M_s$ , conditions (254) may be imposed on the two linearly independent inhomogeneous equations resulting from the application of (252) to both operators (253 a,b). Typically,

$$\begin{aligned} & \langle [f] \alpha(h_1 h_2 h_3) h_1 h_2 (h_1 - h_2); \beta' S' S' T | \mathbb{I}^{(20)}_1; 1_{s'-s} \rangle - \mathbb{I}^{(20)}_2; 1_{s'-s} \rangle / x \\ & \times [f] \alpha(h_1 h_2 h_3) h_1 h_2 (h_1 - h_2); \beta S S T \rangle \end{aligned} \quad (316)$$

$$\begin{aligned} = & \left\{ A_{s's} \langle h_1 h_2 (h_1 - h_2) | C_1^1 - C_2^2 | h_1 h_2 (h_1 - h_2) \rangle \right. \\ & \left. + B_{s's} \langle h_1 h_2 (h_1 - h_2) | G_1^1 - G_2^2 | h_1 h_2 (h_1 - h_2) \rangle \right\} \langle S_1 S S' S' | S' S' \rangle, \end{aligned}$$

and similarly for (253 b); the  $(h_1 h_2 h_3)$  labeling of  $U_3$  is here suppressed for brevity on the right hand side. The operator on the left is by definition (241)

$$\begin{aligned} & \mathbb{I}^{(20)}_1; 1_{s'-s} \rangle - \mathbb{I}^{(20)}_2; 1_{s'-s} \rangle = \\ & \sum_{\sigma \sigma' \tau} \langle \sigma | \hat{U}_{s's} | \sigma' \rangle [C_{1\sigma\tau}^{1\sigma'\tau} - C_{2\sigma\tau}^{2\sigma'\tau}], \end{aligned} \quad (317)$$

and similarly for (253 b); and of course

$$\langle \sigma | A_{S'-S} | \sigma' \rangle = \sqrt{\frac{3}{4}} \langle \frac{1}{2} 1 \sigma' S' - S | \frac{1}{2} \sigma \rangle \quad (318)$$

by the Wigner-Eckhart theorem. Moreover, from expansion (240) we have

$$C_{q\sigma\tau}^{q\sigma'\tau} = \sum_{\mu\mu'} \langle \mu | C_q^q | \mu' \rangle C_{\mu\sigma\tau}^{\mu'\sigma'\tau} \quad (q = 1, \bar{1}, 0 \rightarrow 1, 2, 3) \quad (319)$$

while

$$\langle \mu | C_q^q | \mu' \rangle \equiv \langle n, n, n_0 | C_q^q | n', n', n'_0 \rangle = n'_q \quad (320)$$

from Appendix G. Combining these results carefully one arrives at operators (317) expanded in terms of  $U_{4r}$  generators  $C_{\mu s}^{\mu' s'}$  ( $s, s' = 1, 2, 3, 4$ ) and ( $\mu, \mu' = 1, 2, \dots, 6$  for  $r=6$ ). That is, for  $S'=S=1$  say, (317a) shall be  $\frac{1}{2} [C_{31}^{31} + C_{32}^{32} + C_{33}^{33} + C_{34}^{34}] - \frac{1}{2} [C_{33}^{33} + C_{34}^{34} + C_{51}^{51} + C_{52}^{52}] + [C_{11}^{11} + C_{12}^{12} + C_{43}^{43} + C_{44}^{44}] - [C_{13}^{13} + C_{14}^{14} + C_{41}^{41} + C_{42}^{42}]$ .

The ket states on the left of (316) being of maximum weight in  $U_3$  may be designated briefly as

$$|\alpha f \rangle \alpha(h_1, h_2, h_3) h_1, h_2(h_1, -h_2); \beta S S T \rangle \equiv P_{SS}$$

the first index referring to total intrinsic spin, the second to projection of same. Now, operators  $R_{kl}$  of (180 d) related to cartesian components ( $k, l = 1, 2, 3$ ) can be case in spherical component form:  $R_{qq'} = \frac{1}{4} \sum_{\sigma\tau\sigma'\tau'} (M_q)_{\sigma\tau}^{\sigma'\tau'} (N_{q'})_{\sigma'\tau'}^{\sigma\tau} C_{\sigma\tau}^{\sigma'\tau'}$

where the Pauli matrices now in spherical components are

$$M_1 = -\sqrt{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad M_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad M_{-1} = \sqrt{2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with rows and columns given by  $\sigma, \sigma' = \frac{1}{2}, -\frac{1}{2}$  and similarly for  $N_q$  ( $q = 1, 0, -1$ ) with rows and columns given by

$\tau, \tau' = \frac{1}{2}, -\frac{1}{2}$ . The operator  $2R_{T0}$  is then

$$2R_{T0} = \frac{1}{\sqrt{2}} (C_3' - C_4^2)$$

in terms of  $U_4$  lowering generators belonging to the set (61). It is the lowering operator for total intrinsic spin  $S$ , i.e.,

$$2R_{T0} \mathbb{P}_{SS} = \mathbb{P}_{S-1, S-1}$$

and moreover keeps the result in maximum projection. Likewise,  $2R_{0T}$  would be the lowering operator for total isospin  $T$ . The case of  $F^{20}$  involves  $(h_1 h_2 h_3) = (710)$  and from Table VI.3.1 for  $S = 1$  is simply

$$\mathbb{P}_{11} = \frac{1}{6} (2, 111) \equiv b_{21}^+ b_{11}^+ b_{12}^+ b_{13}^+$$

and consequently

$$\mathbb{P}_{00} = 2R_{T0} \mathbb{P}_{11} = \frac{1}{\sqrt{2}} (b_{23}^+ b_{11}^+ b_{12}^+ b_{13}^+ - b_{21}^+ b_{11}^+ b_{14}^+ b_{13}^+).$$

Evaluation of the left-hand members of equations (316) now follows easily. On the right hand side we have, from (144 a),

$$\langle h_1 h_2, (h_1 - h_2) | C_1' - C_2^2 | h_1 h_2, (h_1 - h_2) \rangle = h_1 - h_2$$

and similarly for  $C_2^1 - C_3^2$ . Furthermore, by definition (249) and condition (144 b)  $G_1' - G_2^2 = (C_1')^2 - (C_2^2)^2 + C_1' - C_2^2$



and similarly for  $\mathcal{G}_2^2 - \mathcal{G}_3^2$ , so that

$$\langle h_1 h_2, (h_1 - h_2) | \mathcal{G}_1^1 - \mathcal{G}_2^2 | h_1 h_2, (h_1 - h_2) \rangle = h_1^2 - h_2^2 + h_1 - h_2$$

and likewise for  $\mathcal{G}_2^2 - \mathcal{G}_3^2$ . Finally, for  $(h_1 h_2 h_3) = (710)$  and  $S' = S = 1$ , say, equations (316) reduce to

$$1 = (6 A_{11} + 60 B_{11}) \frac{1}{\sqrt{2}} \quad \frac{1}{2} = (A_{11} + 2 B_{11}) \frac{1}{\sqrt{2}}$$

from which follow  $A_{11}$  and  $B_{11}$  numerically. The cases  $(S', S) = (1, 0)$  and  $(0, 1)$  then involve use of operator  $2R_{T0}$  as indicated before and the complete results are:

TABLE VIII.1

$F^{20}$   
 $(k_1 k_2) = (71)$

S'S	11	10	01
$A_{S'S}$	$7\sqrt{2}/12$	$-2/3$	$2\sqrt{3}/3$
$B_{S'S}$	$-\sqrt{2}/24$	$1/12$	$-\sqrt{3}/12$

Notice that the case  $S = S' = 0$  does not appear as then matrix element (263) vanishes, since the factor  $\langle S1 M_S q'' | S'1 M_S' \rangle$  would be zero by violation of triangularity. This reflects the fact that spin-orbit interaction is zero in first-order for singlet states.

To compute reduced matrix elements  $\langle (k_1 k_2) \omega' L' || \mathcal{K} || (k_1 k_2) \omega L \rangle$  recall the second expression for it in (258) and algebraic formula (262). Transformation coefficients  $\langle q_1 q_2 (k_1 k_2 M_L = 1) \omega L \rangle$  form the matrix diagonalizing  $\| \langle (k_1 k_2) q_1' q_2' M_L | \mathcal{L}^2 | (k_1 k_2) q_1 q_2 M_L \rangle \|$  for  $M_L = 1$ , i.e., for  $(k_1 k_2) = (71)$  the former matrix is (I.9). Thus, one begins by constructing  $\| \langle (k_1 k_2) q_1' q_2' M_L | \mathcal{K}_0^{(1)} | (k_1 k_2) q_1 q_2 M_L \rangle \|$

for  $(k_1, k_2) = (7, 1)$  and  $M_L = 1$  by formula (262) which shows that our matrix will have many zeros. In accordance with (258), the result is then transformed by similarity with matrix (I.9) to give

$\| \langle (7, 1) L' 1 | \mathcal{K}_0^{(1)} | (7, 1) L 1 \rangle \|$  which has non-zero elements only along the principal diagonal and the ones contiguous to this above and below. This results from  $\hat{L}^{\dagger} = \hat{L} + 1$  which implies  $L^{\dagger} = L, L \pm 1$ . Each element is then divided by the appropriate  $\langle L 1 1 0 | L' 1 \rangle$  coefficient and the final result is:

TABLE VIII.2

MATRIX  $\| \langle (k, k_2) \omega' L' \| \mathcal{K}_0^{(1)} \| (k, k_2) \omega L \rangle \|$  for  $(k, k_2) = (7, 1), \omega = \omega'$

$L' \backslash L$	1	2	3	4	5	6	7
1	$5\sqrt{2}$	$-6\sqrt{2/3}$					
2	$6\sqrt{2/5}$	$\frac{26\sqrt{6}}{3}$	$\frac{8}{3}\sqrt{\frac{231}{35}}$				
3		$-\frac{8}{3}\sqrt{\frac{33}{7}}$	$\frac{50}{\sqrt{3}}$	$-30\sqrt{\frac{2}{105}}$			
4			$10\sqrt{\frac{2}{15}}$	$\frac{96}{\sqrt{5}}$	$-\frac{\sqrt{55 \times 1307}}{36}$		
5				$\frac{\sqrt{5 \cdot 1307}}{12}$	$131\sqrt{\frac{2}{15}}$	$\frac{1}{75}\sqrt{\frac{25699 \cdot 13}{7}}$	
6					$\frac{-1}{75}\sqrt{\frac{25699 \cdot 11}{87}}$	$206\sqrt{\frac{2}{21}}$	$-\frac{1}{4}\sqrt{\frac{897 \cdot 35}{31}}$
7						$-\frac{1}{4}\sqrt{\frac{299 \cdot 91}{31}}$	$124\sqrt{\frac{2}{7}}$

In conclusion, formula (263) can now be used to construct the matrix of spin-orbit interaction  $\mathcal{W}_{s.o.}^p$  for a given  $SU_3$  representation, wherein  $\omega = \omega'$  automatically, the results for  $F^{20}$  with  $(k_1 k_2) = (71)$  are thus:

$J=0$

LS	11
	$-3/4$

$J=1$

LS	10	11	21
	0	$-\sqrt{2}/4$	$-1/\sqrt{6}$
		$-3/8$	$\sqrt{3}/12$
			$-2/3$

$J=8$

LS	71
	1.5

$J=2$

LS	20	11	21	31
	0	$1/\sqrt{10}$	$1/3\sqrt{6}$	$\frac{2}{3}\sqrt{\frac{11}{15}}$
		$3/8$	$3/4\sqrt{15}$	0
			$-2/9$	$-\frac{1}{9}\sqrt{\frac{22}{5}}$
				$-17/18$

$J=3$

LS	30	21	31	41
	0	$-\frac{2}{9}\sqrt{\frac{33}{7}}$	$\frac{1}{6\sqrt{3}}$	$-\sqrt{\frac{5}{42}}$
		$4/9$	$-\frac{2}{9}\sqrt{\frac{11}{7}}$	0
			$-17/72$	$\frac{1}{4}\sqrt{\frac{5}{14}}$
				$-11/12$

$J=4$

LS	40	31	41	51
	0	$\frac{5}{6}\sqrt{\frac{2}{15}}$	$\frac{4\sqrt{5}}{15}$	$-\frac{\sqrt{55 \cdot 307}}{12 \cdot 36}$
		$\frac{17}{24}$	$\frac{5}{12\sqrt{6}}$	0
			$-\frac{11}{60}$	$\frac{\sqrt{11 \cdot 1307}}{12 \cdot 36}$
				$-\frac{79}{60}$

$J=5$

LS	50	41	51	61
	0	$\frac{\sqrt{5 \cdot 1307}}{144}$	$\frac{11}{12}\sqrt{\frac{2}{15}}$	$\frac{\sqrt{25699 \cdot 13}}{12 \cdot 75\sqrt{7}}$
		$11/15$	$\frac{\sqrt{7842}}{288}$	0
			$-79/360$	$\frac{\sqrt{390 \cdot 25699}}{6 \cdot 24 \cdot 75\sqrt{7}}$
				$-11/9$

$J=6$

LS	60	51	61	71
	0	$\frac{\sqrt{25699 \cdot 11}}{12 \cdot 75\sqrt{7}}$	$\frac{19}{9}\sqrt{\frac{3}{14}}$	$\frac{\sqrt{35 \cdot 897}}{48\sqrt{31}}$
		$79/72$	$\frac{\sqrt{66 \cdot 25699}}{144 \cdot 75}$	0
			$-11/63$	$-\frac{\sqrt{26910}}{96\sqrt{31}}$
				$-12/7$

$J=7$

LS	70	61	71
	0	$-\frac{\sqrt{299 \cdot 91}}{48\sqrt{31}}$	$\sqrt{2/7}$
		$\frac{22}{21}$	$-\frac{\sqrt{26 \cdot 299}}{48\sqrt{31}}$
			$-3/14$

# IX. PREDICTED LOW-LYING LEVEL STRUCTURE OF $F^{20}$ .

The interaction model to be used was discussed in Chapter IV with special emphasis on their group theoretical symmetries. It should be kept in mind that  $P$  refers to orbital pairing as explained in IV. 3., and  $Q^2$  to orbital quadrupole-quadrupole ---- extension of the former to include spin might provide a more realistic type of pairing but would introduce the symplectic group  $Sp_6$  instead of  $R_6$  and force us to use a  $jj$  coupling base. A linear combination of our  $P$  and  $Q^2$  may be used as a model for central two-body residual forces between extra-closed-shell nucleons and evidence in support of this was cited at the end of IV.3. Regarding exchange effects, an exchange operator  $A$  taking a simple form at long ranges as discussed in IV.2 is employed with the  $Q^2$  portion of the model which approximates long range. Exchange at short ranges, i.e., in association with  $P$ , is neglected under the assumption of predominance of the Wigner component (into which the Majorana collapses at zero range) over the Bartlett component (into which the Heisenberg collapses) ---- for a Rosenfeld mixture the relative intensities are 80 to 20% respectively. To this is then added the single-body spin-orbit interaction discussed in IV.6. Our total interaction hamiltonian is stated by (264) or, parametrically more convenient, by (265) and is acting amongst the four extra-closed-shell nucleons only. The doubly magic  ${}_2O_8^{16}$  core is presupposed spherical and inert though departure from this simple state of affairs is hopefully expected to be simulated at least approximately by the long-ranged  $Q^2$  interaction between extra-closed-shell particles.

The lowest levels of  $F^{20}$  will be given by  $M_T = T = 1$ . From (A.4) the most antisymmetric spin-isospin Young partition which by Wigner<sup>47)</sup> is lowest in energy is  $[f] = [211]$  to which, for  $T = 1$ , are associated  $S=0$  and 1. The corresponding orbital symmetry is then  $[f] = [31]$  whose lowest-energy  $SU_3$  irreducible representation by Table V.1.1/<sup>18</sup> $(k_1, k_2) = (71)$ . This representation, to which our set of basis functions is restricted (Elliott hypothesis<sup>12)</sup>), contains L-values equal to 1, 2, 3, 4, 5, 6 and 7. The resulting J-values are thus 0, 1<sup>3</sup>, 2<sup>4</sup>, 3<sup>4</sup>, 4<sup>4</sup>, 5<sup>4</sup>, 6<sup>4</sup>, 7<sup>3</sup> and 8 so that our largest matrix for *Hint* is 4 X 4. The base is designated by

$$|[f] \alpha'(k, k_2) \omega L; \beta' S; J T\rangle \left\{ \begin{array}{l} [f](k, k_2) = [31](71) \\ \alpha' \text{ unneeded for } [f] = [31] \\ \omega' \quad " \quad " \quad (k, k_2) = (71) \\ \beta' \quad " \quad " \quad [f'] = [211] \\ M_J = J \text{ and } M_T = T \end{array} \right.$$

The (diagonal) matrices of  $\mathcal{Q}^2$  in this base are given by eigenvalues (176) for  $\mathcal{Q}^2$  and (186) for exchange  $\mathcal{Q}$ , in accordance with (187). The matrices for  $\mathcal{P}$ , which are also diagonal in the orbital angular momentum quantum number L, are composed of pairing eigenvalues (314) in the manner of (315). Finally, the matrices of  $\mathcal{W}_{30}$  are given at the end of the previous chapter and their construction was discussed there.

The matrices were fed into an IBM 1620 computer for diagonalization by the Jacobi method, with parameters  $\alpha$  and  $\beta$  of (265) varied.

Scanty and rather ambiguous information on the spins of low-lying  $F^{20}$  is known to date. The  $\beta$ -decay<sup>79)</sup> from its ground state to the second excited  $Ne^{20}$  level  $J^\pi = 2^+$  suggests a  $J^\pi = 2^+$  spin for the  $F^{20}$  ground state. Considerations<sup>80)</sup> involving the Nilsson rotational model corroborate this assignment. The  $\beta$ -decay from the  $O^{20}$  ground state  $J^\pi=0^+$  to the 1.064 Mev level of  $F^{20}$  suggests a  $J^\pi=1^+$  spin<sup>81)</sup>. Mazari and co-workers<sup>82)</sup> have studied the  $F^{19}$  ( $d, p\gamma$ )  $F^{20}$  stripping reaction measuring gamma-ray energies for 15 excited states as well as angular distributions of proton groups. Distorted-wave theory fits to the data permitted generally reliable neutron-capture angular momenta  $l_n$  assignments (see following Figure). The same reaction was studied by Chagnon<sup>83)</sup> who measured ( $p, \gamma$ )-directional correlations<sup>85)</sup> through four low-lying excited states. These results were interpreted by him via DWBA computations, thus providing him further restrictions on possible spin values for some of these levels. To the 0.66 level he assigned a  $J^\pi=3^+$ , but a  $2^+$  is not inconsistent, and suggests that separate measurement of the mixing ratio for this transition would suffice to determine the spin uniquely. (Based on the stripping reduced width, Dazai<sup>84)</sup> previously obtained a  $J^\pi=3^+$  for this level.) For the 0.989 level the angular distribution correlation is isotropic and this a  $0^+$  assignment would not be inconsistent. Chagnon finds a  $1^+$  for the 1.064 level admissible, but was unable to discard  $2^+$  or  $3^+$ . The 1.312 level he finds decaying to the ground state with an intensity ratio of  $E2 / \text{total} > 0.97$  leaving little doubt of a  $J^\pi=2^+$ . These results are summarized on the extreme left of the Figure, all levels have  $T=1$  and  $T=2$  levels are expected to begin appearing in the neighborhood of as high as 6 Mev.

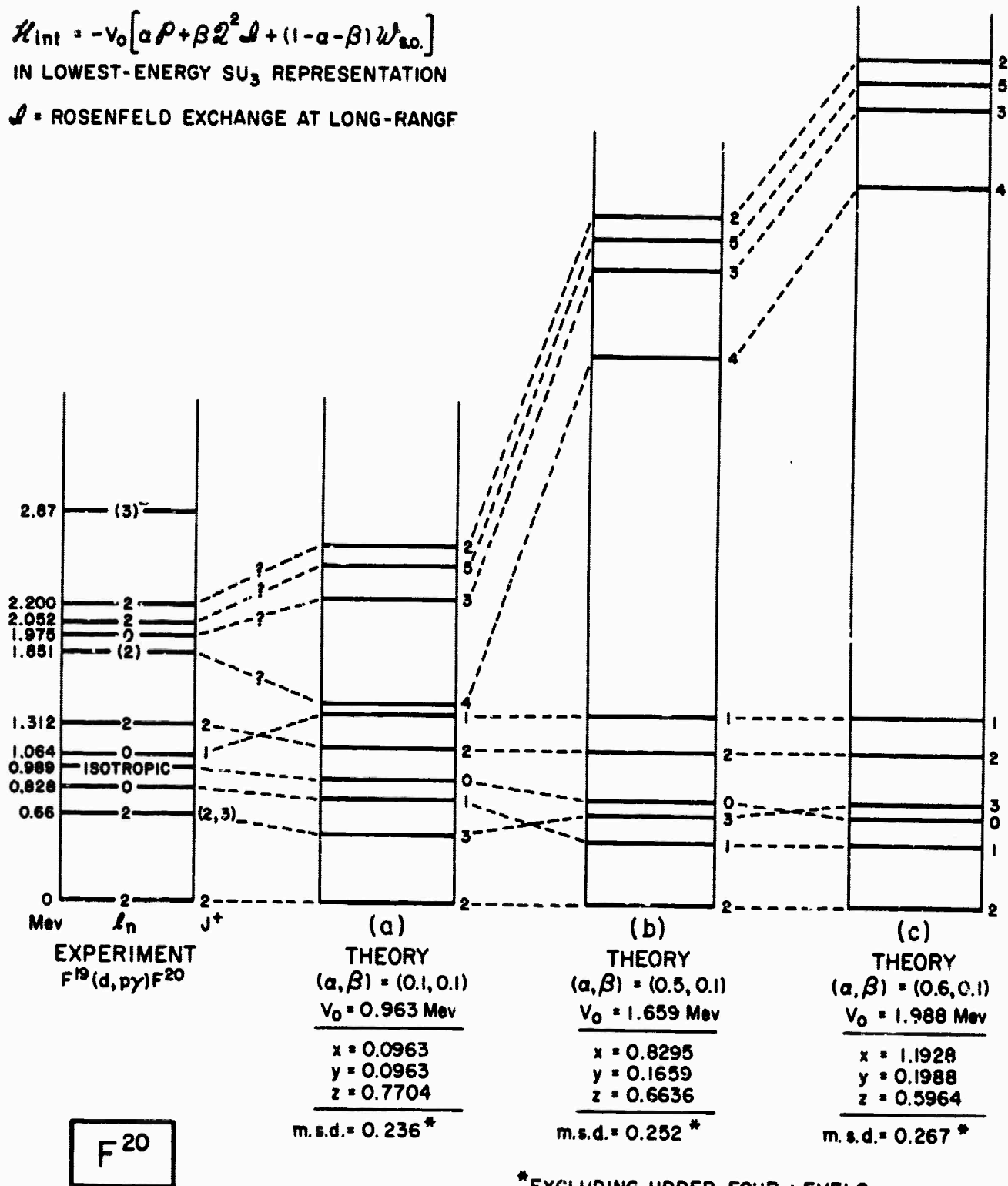
Variation of parameters  $\alpha$  and  $\beta$  by tenths ( $\alpha+\beta\leq 1$ ) in the computer diagonalization program results in a set of 66 predicted spectra. The non-degenerate ones fall into three groups of level-order  $J=2, 3, 1, 0, 2, 1, \dots$  and  $2, 1, 3, 0, 2, 1, \dots$  and  $2, 1, 0, 3, 2, 1, \dots$ . The first and second  $J=2$  predicted levels were associated with the empirical ground and 1.312 levels, respectively, and the second  $J=1$  predicted level related to the 1.064 empirical level. The remaining predicted levels, i.e.,  $J=3, 0$  and first 1 were associated with remaining empirical levels in order of appearance and least-mean-squares (involving above  $J$ -levels only) calculated for the three groups. The over-all intensity parameter  $V_0$  was found from the least-mean-squares procedure. These results are displayed in the Figure to the right of the empirical spectrum. Parameters  $x, y$ , and  $z$  given by  $x = V_0\alpha$ ,  $y = V_0\beta$  and  $z = V_0(1-\alpha-\beta)$  denote the relative intensities, respectively, of pairing (short-range), quadrupole-quadrupole (long-range) and spin-orbit interactions.

In conclusion we may state the following. The spectrum in optimum agreement by least-mean-squares (a in Figure) predicts a  $J=3$  for the first excited level, followed by a  $J=1$  and  $J=0$  levels. The first  $J=4$  level is excessively depressed here. However, adding a slight amount more of pairing (moving from a to b) raises it considerably. Our most encouraging result is perhaps the reproduction of the gross characteristics of the empirical spectrum at low energies, i.e., at some point slightly away from (a) and toward (b) one obtains a "gap" above the ground state followed by a group of 5 levels, followed in turn by another gap above which is another group of levels. For all  $(\alpha, \beta)$  parameter values allowed, the second  $J=2$

$$\mathcal{H}_{\text{int}} = -V_0 [a\rho + \beta \mathcal{Q}^2 \mathcal{J} + (1-a-\beta) \mathcal{W}_{\text{so}}]$$

IN LOWEST-ENERGY  $SU_3$  REPRESENTATION

$\mathcal{J}$  = ROSENFELD EXCHANGE AT LONG-RANGE





appeared below the second  $J=1$  ---- in seeming discrepancy with experiment. A similar situation arose from this model for the second  $J=2$  level of  $F^{18}$  for a wide range of the parameters<sup>86)</sup> (with and without the restriction of the base to the lowest-energy  $SU_3$  representation). More definite experimental information on spins is required, in particular for the first three excited states, to render a more complete test of these results. Explicit wave functions for the low-lying states of spectrum (a) will be available in a forthcoming paper.<sup>86)</sup>

## X. SUMMARY AND CONCLUSIONS.

Moshinsky's approach by group theoretical techniques to the nuclear shell many-body problem is presented in general form and illustrated with a calculation of the low-lying energy levels of Fluorine-20. On comparison with experiment, encouraging results are found but more definitive conclusions regarding detailed agreement must await further empirical spin assignments for this nucleus.

Every low-energy nuclear level structure calculation is beset by two basic difficulties requiring basic approximations of one sort or another: uncertainties with regard to the nuclear force and those regarding the number of nucleon configurations to be considered.

The customary use of a reasonably shaped two-body residual interaction potential well is justified only "a posteriori" on phenomenological grounds but not on fundamental theory. Therefore, an equally phenomenological model hamiltonian consisting of a linear combination of orbital pairing, quadrupole-quadrupole (with exchange character) and single-body spin-orbit interactions was employed. The first and second interactions respectfully approximate the short- and long-ranged correlations of the central two-nucleon residual interaction. The advantages of this model lie in the fact that its various portions possess group symmetries of considerable convenience in the calculation of matrices.

The enormous number of totally antisymmetric states arising from all possible particle configurations was limited to a smaller, more feasible number by making the following restrictive assumptions. (1) Only configurations arising from a single mayor

oscillator shell (the 2s-1d shell) are considered so that only positive parity states will result. (2) Assuming with Wigner that the residual two-nucleon forces inside the nucleus are attractive and to a large extent independent of the nucleon spins and charges, one may further restrict the number of states remaining in (1) to those corresponding to the most symmetric partition under permutation of the space variables, compatible with the total isospin  $T$  of the lowest energy for the given nucleus. (3) Accepting on a tentative basis the proposal of Elliott's that the lowest-energy  $SU_3$  (group of unitary unimodular transformations in 3 dimensions) representation contained in the partition chosen by (2) will to a reasonable degree determine the low-lying states for nuclei in the 2s-1d shell, one further delimits the number of states to be used in calculating the energy matrix. Whereas the first two restrictive assumptions are usual in shell model calculations, the third is more group theoretical in character and is based on Elliott's work showing that  $SU_3$  provides a link between the shell and collective models. For the simpler nuclear p-shell (lying immediately below our 2s-1d shell) this group theoretic classification scheme by  $SU_3$  collapses into the ordinary LS coupling scheme. There is moreover no conclusive evidence that  $SU_3$  cannot serve usefully in dealing with more complex nuclear shells beyond the 2s-1d shell.

The language adopted for operators as well as state-vectors is the second-quantization formalism involving creation and annihilation operators for fermions. Thus, a given problem becomes more transparent to possible group symmetries that may prevail in a given operator or which are to be "built-into" a given state-vector. Consequently, the computation of matrix elements is reduced to simple commutator algebra.

The construction of N-particle state-vectors possessing the specific quantum numbers demanded by a given problem was thus approached group theoretically. These techniques rest on the studies of S. Lie and E. Cartan in the theory of continuous groups of transformations and are essentially generalizations of angular momentum theory to symmetries higher than rotational. They are discussed in as strict analogy as possible to the better-known concepts of angular momentum and its deduced consequences.

For nuclei with a number of extra-closed-shell nucleons  $\geq 4$  the present methods are felt to be an improvement over conventional fractional parentage methods which then become particularly cumbersome. The results obtained thus far in this work with these methods are considered to justify further work along these lines for a larger number of nuclei. This program is being pursued at the University of Mexico where an exhaustive study of 2s-1d shell nuclei, their energies, moments, transition rates and other low-energy properties is in progress.

# APPENDIX A

## SUPERMULTIPLETS OF A GIVEN SYMMETRY PATTERN IN THE 2s-1d SHELL.

Derivation of the irreducible representation labels of  $R_3$  contained in a given one of  $U_p$  has been discussed, among others, by Jahn<sup>41)</sup> who considers  $N$  particles in the p shell ( $r=3$ ) and in the d shell ( $r=5$ ). In the 2s-1d shell  $r=6$ . A single particle within this shell has only one possible Young partition, namely,  $[1] = \square$  and since s means  $l=0$  and d means  $l=2$ , one has

$$U_6 \supset R_3$$

$$N=1: \quad \square \supset l=0,2$$

For two particles one can form the outer product of two one-particle  $U_6$  representations and according to Littlewood's rules<sup>72, 34)</sup> get

$$\square \otimes \square = \square\square + \square$$

i.e., the symmetric  $[2]$  and the antisymmetric  $[11]$  irreducible representations of  $U_6$ . Defining  $\Delta_{\mu_1}^{s_1} \equiv b_{\mu_1, s_1}^+$  and

$$\Delta_{\mu_1, \mu_2}^{s_1, s_2} \equiv b_{\mu_1, s_1}^+ b_{\mu_2, s_2}^+ - b_{\mu_1, s_2}^+ b_{\mu_2, s_1}^+$$

which is obviously symmetric under interchange of orbital coordinates (the indices  $\mu$ ); two particles in the 2s-1d shell give rise the following orbital configurations with associated angular momenta and

wave functions in second-quantization form:

$$\begin{array}{c|c|c}
 \frac{(s)^2}{L=0} & \frac{(s)^1(d)^1}{L=2^2} & \frac{(d)^2}{L=0,1,2,3,4} \\
 \Delta_{00,00}^{S_1 S_2} & \Delta_{00,2m}^{S_1 S_2} ; \Delta_{00}^{S_1} \Delta_{2m}^{S_2} & \sum_{m_1 m_2} \langle 22 m_1 m_2 | L M_L \rangle \Delta_{2m_1, 2m_2}^{S_1 S_2} \\
 & & = (-)^L \sum_{m_1 m_2} \langle 22 m_1 m_2 | L M_L \rangle \Delta_{2m_1, 2m_2}^{S_1 S_2}
 \end{array} \quad (A.1)$$

where  $\mu_i \equiv \nu_i l_i m_i$  and as  $\nu_i$  refers always to the same shell, it is suppressed.

In configuration  $(d)^2$ , the function is symmetric for  $L=0,2,4$  and antisymmetric for  $L=1,3$ . Hence, the total  $L$ -structures of the symmetric (  $\square$  ) and antisymmetric (  $\boxminus$  ) partitions are;

$$\begin{array}{l}
 \square > L = 0^2, 2^2, 4 \\
 \boxminus > L = 1, 2, 3
 \end{array} \quad (A.2)$$

Now, the 3-particle antisymmetric partition is  $[111] \equiv \boxminus$  with wave function  $\Delta_{\mu_1 \mu_2 \mu_3}^{S_1 S_2 S_3}$ , where  $\mu_i = l_i m_i$  ( $i=1,2,3$ ) and  $m_i = 2,1,0,0,-2$  since  $l_i = 0,2$ . The total (positive)  $M$  values can be formed in the following ways:

$$\left. \begin{array}{l} 2+1+0 \\ 2+1+0' \end{array} \right\} = 3 \quad \left. \begin{array}{l} 2+1-1 \\ 2+0+0' \end{array} \right\} = 2 \quad \left. \begin{array}{l} 2+1-2 \\ 2+0-1 \\ 2+0'-1 \\ 1+0+0' \end{array} \right\} = 1 \quad \left. \begin{array}{l} 2+0-2 \\ 2+0'-2 \\ 1+0-1 \\ 1+0'-1 \end{array} \right\} = 0$$

Hence,

$$M = \begin{array}{cccc} \circ & \circ & \circ & \circ \\ 1 & 1 & 1 & 1 \\ & 2 & 2 & \\ & 3 & 3 & \end{array}$$

so that  $L=1^2, 3^2$  and thus, in the 2s-1d shell

$$\boxed{\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}} > L = 1^2, 3^2.$$

Knowing the L-structure of  $\mathcal{U}_6$  partitions  $[1]$ ,  $[2]$ ,  $[11]$  and  $[111]$ , one may proceed to deduce the rest by a chain calculation:

$$\boxed{\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}} \otimes \square = \boxed{\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}} + \boxed{\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}}$$

will have an L-structure given by the vectorial sum of L's

$$1, 2, 3 + 0, 2 = 0, 1^4, 2^4, 3^4, 4^2, 5$$

from which, if we subtract the  $\boxed{\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}}$  L-structure one obtains

$$\boxed{\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}} > L = 0, 1^2, 2^4, 3^2, 4^2, 5.$$

Continuing,

$$\square \otimes \square = \square + \square$$

has L-structure

$$0^2, 2^2, 4 + 0, 2 = 0^4, 1^2, 2^7, 3^3, 4^4, 5, 6$$

which leaves, after subtraction of the  $\square$  L-structure,

$$\square > L = 0^3, 2^3, 3, 4^2, 6.$$

The partition  $[1111]$  has the same L-structure as  $[11]$ , for  $U_6$ .

Proceeding with the chain calculation sketched here one further deduces

$[f]$	L-structure
$[211]$	$1^5, 2^3, 3^5, 4^2, 5^2$
$[22]$	$0^3, 1, 2^5, 3^2, 4^4, 5, 6$
$[31]$	$0^2, 1^4, 2^7, 3^6, 4^5, 5^3, 6^2, 7$
$[4]$	$0^4, 2^5, 3, 4^4, 5, 6^2, 8$
$\vdots$	

(A.3)

To find the (S,T)-structure (or multiplet structure) of a given Young partition  $[f]$  one must effect the reduction

$$U_4 > U_2 \times U_2$$



the irreducible representations of  $U_4$  being  $[\tilde{f}]$ . A chain calculation similar to the previous reduction  $U_6 \supset R_3$  is again useful; one starts with the following primitive cases:

The  $[1111]$  representation of  $U_4$  is equivalent to  $[0]$  which contains  $(S,T) = (0,0)$ . Representation  $[1]$  of  $U_4$ , explicitly reduced into those of  $U_2 \times U_2$  is

$$\square \supset \square \times \square$$

the  $\times$  symbol standing for the inner product. Thus

$$\square \text{ of } U_4 \supset (S,T) = (\frac{1}{2}, \frac{1}{2}).$$

Representations  $[2]$  and  $[11]$  of  $U_4$  are formed by

$$\begin{aligned} \square\square &\supset \square\square \times \square\square & \square\square &\supset \square\square \times \square\square \\ &\supset \square\square \times \square\square & &\supset \square\square \times \square\square \end{aligned}$$

that is, the symmetric spin-isospin two-particle function is either spin symmetric times isospin antisymmetric or spin antisymmetric times isospin antisymmetric; the antisymmetric spin-isospin function is either spin symmetric times isospin antisymmetric or spin antisymmetric times isospin symmetric. The corresponding  $(S,T)$  structures are thus

$$\square\square \text{ of } U_4 \supset (S,T) = (1,1), (0,0)$$

$$\square \text{ of } U_4 \supset (S,T) = (1,0), (0,1).$$

Tabulating, one has

$U_4$	$U_2 \times U_2$	$(S, T)$
$[1111] = [0]$	$[0] \times [0]$	$(0,0)$
$[1]$	$\square \times \square$	$(\frac{1}{2}, \frac{1}{2})$
$[2]$	$\square \times \square ; \theta \times \theta$	$(1,1) \quad (0,0)$
$[11]$	$\square \times \theta ; \theta \times \square$	$(1,0) \quad (0,1)$

Then, in  $U_4$  one uses Littlewood's rules for external products to form

$$\square \otimes \square = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}$$

whose  $(S,T)$ -structure is found by vector addition of  $S$  and  $T$  separately,

$$\begin{array}{c} (1,1) \\ (0,0) \end{array} + \begin{array}{c} (\frac{1}{2}, \frac{1}{2}) \\ (\frac{1}{2}, \frac{1}{2}) \end{array} = \begin{array}{c} (\frac{1}{2}, \frac{1}{2})^2 \\ (\frac{1}{2}, \frac{3}{2}) \end{array} \begin{array}{c} (\frac{3}{2}, \frac{1}{2}) \\ (\frac{3}{2}, \frac{3}{2}) \end{array}$$

Again, in  $U_4$

$$\theta \otimes \square = \begin{array}{|c|} \hline \theta \\ \hline \theta \\ \hline \end{array} + \begin{array}{|c|} \hline \theta \\ \hline \theta \\ \hline \end{array} = \begin{array}{|c|} \hline \square \\ \hline \theta \\ \hline \end{array} + \begin{array}{|c|c|} \hline \theta & \theta \\ \hline \end{array},$$

since  $[111] \equiv [1]$ , has  $(S,T)$ -structure

$$\begin{array}{c} (1,0) \\ (0,1) \end{array} + \begin{array}{c} (\frac{1}{2}, \frac{1}{2}) \\ (\frac{1}{2}, \frac{1}{2}) \end{array} = \begin{array}{c} (\frac{1}{2}, \frac{1}{2})^2 \\ (\frac{1}{2}, \frac{3}{2}) \end{array} \begin{array}{c} (\frac{3}{2}, \frac{1}{2}) \\ (\frac{3}{2}, \frac{1}{2}) \end{array}$$

which upon subtracting that of  $[1]$  leaves

$$\begin{array}{|c|} \hline \theta \\ \hline \theta \\ \hline \end{array} \text{ of } U_4 > (S,T) = \begin{array}{c} (\frac{1}{2}, \frac{1}{2}) \\ (\frac{1}{2}, \frac{3}{2}) \end{array} \begin{array}{c} (\frac{3}{2}, \frac{1}{2}) \\ (\frac{3}{2}, \frac{1}{2}) \end{array}$$

Subtracting this from the multiplet structure of  $[21] + [3]$  above one gets

$$\square \text{ of } U_4 \supset (S,T) = \left(\frac{1}{2}, \frac{1}{2}\right) \left(\frac{3}{2}, \frac{3}{2}\right).$$

The calculation can be continued nad extensive tables prepared ----- Jahn and Hammermesh (refs. 34 and 41 ) have rather complete ones. We extract from those sources the table for  $N=4$  particles as it will concern us directly.

N	$U_4: [\tilde{f}]$	$(S,T)$ -structure
4	$[1111]$	$(0,0)$
	$[211]$	$(1,0) (0,1) (1,1)$
	$[22]$	$(0,0) (0,2) (2,0) (1,1)$
	$[31]$	$(0,1) (1,0) (1,1) (1,2) (2,1)$
	$[4]$	$(0,0) (1,1) (2,2)$

(A.4)

If multiple  $(S,T)$  values occur for a given  $[\tilde{f}]$  -----does not happen for  $N \leq 5$  ----- the additional label  $\beta$  in chain (99) is required.<sup>46)</sup>

APPENDIX B.

UNIQUENESS OF TOTALLY ANTISYMMETRIC MANY-FERMION POLYNOMIAL OF  
MAXIMUM WEIGHT.

A general many-particle eigenpolynomial is given by (21), namely

$$P = \sum_{\mu_1 S_1, \mu_2 S_2, \dots, \mu_N S_N} B_{\mu_1 S_1, \mu_2 S_2, \dots, \mu_N S_N} b_{\mu_1 S_1}^+ b_{\mu_2 S_2}^+ \dots b_{\mu_N S_N}^+.$$

For a given set of values  $S_1, S_2, \dots, S_N$  the corresponding term can be chosen to transform irreducibly according to the permutation group

$\Pi_N$  as  $[h_1, h_2, \dots, h_N]$ . Likewise, for a given set  $\mu_1, \mu_2, \dots, \mu_N$  one can choose the corresponding result to transform irreducibly as

$[v_1, v_2, v_3, v_4]$  of the same  $\Pi_N$  group. Due to Weyl's theorem,<sup>43)</sup> a function transforming irreducibly under  $\Pi_N$  as  $[h_1, h_2, \dots, h_t]$  will also transform so under  $\mathcal{U}_t$  (where  $t$  is the dimensionality of the coordinate space), and the irreducible representations of the latter are characterized by the same label  $[h_1, h_2, \dots, h_t]$ . Therefore, the (maximum weight) representations  $[h_1, h_2, \dots, h_r]$  of  $\mathcal{U}_r$  and

$\{v_1, v_2, v_3, v_4\}$  of  $\mathcal{U}_4$ , specified by (73) and (74), are irreducible.

The Young pattern of  $\Pi_N$  for permutations in the total space of  $4r$  dimensions is  $[1^N 0^{4r-N}]$ ; this is also the irreducible representation of  $\mathcal{U}_{4r}$ . And since

$$\mathcal{U}_{4r} > \mathcal{U}_r \times \mathcal{U}_4$$

then  $[1^N 0^{4r-N}] > [h_1 h_2 \dots h_r] \times \{V_1 V_2 V_3 V_4\}$

where, as in example (80),  $[h_1 h_2 \dots h_r] = \overline{[V_1 V_2 V_3 V_4]}$

Simple examples follow:

N= 1 particle:  $[1] > [1] \times [1]$  (a)

N= 2 particles  $[11] > [2] \times [11]$  (b)

N= 3 particles  $[111] > [3] \times [111]$  (c)

$> [21] \times [21]$  (d)

N= 4 particles  $[1111] > [4] \times [1111]$  (e)

$> [31] \times [211]$  (f)

$> [22] \times [22]$  (g)

N particles

$[1^N] \equiv \left\{ \begin{array}{c} \boxed{\phantom{0}} \\ \vdots \\ \boxed{\phantom{0}} \end{array} \right\} \begin{array}{c} \text{N} \\ \text{blocks} \end{array} > \left\{ \begin{array}{c} \boxed{\phantom{0}} \boxed{\phantom{0}} \boxed{\phantom{0}} \\ \vdots \\ \boxed{\phantom{0}} \end{array} \right\} \begin{array}{c} \text{(rows } \leq r) \end{array} \times \left\{ \begin{array}{c} \boxed{\phantom{0}} \boxed{\phantom{0}} \boxed{\phantom{0}} \dots \end{array} \right\} \quad (h)$

In each case, the orbital and spin-isospin permutation symmetries must be such that a totally antisymmetric orbit-spin-isopin permutation pattern results. In (a), (b), (c) and (e) we have examples of "pure symmetries", i.e., symmetric x antisymmetric = antisymmetric. In all other examples "mixed symmetries" such as [21] are involved,

but the net result is allowed by the Pauli principle.

We have shown in (79) how to construct the function which is of maximum weight in the representation of the subgroup  $U_r \times U_4$  contained in the  $[1^N 0^{4r-N}]$  representation of  $U_{4r}$ . This polynomial is unique as otherwise it would imply that with a given representation  $[h_1, h_2, \dots, h_r]$  we can associate besides the conjugate  $\{v_1, v_2, v_3, v_4\}$  other representations of  $U_4$  and still get the totally antisymmetric representation of  $U_{4r}$ . However, it is shown by Bayman<sup>73)</sup> that this is not possible for representations of the permutation group  $\Pi_N$  which are closely connected with those of the unitary groups.

Conclusion. Polynomial  $\mathbb{P}$  as a solution of eqns. (73) and (74) is unique.

APPENDIX C.

ANGULAR MOMENTUM COMPONENTS AS GENERATORS OF THE GROUP  $R_3$ .

A real infinitesimal orthogonal transformation of the cartesian components is simply

$$x'_i = \sum_j R_{ij} x_j \quad (i, j = 1, 2, 3); \quad R = I + \epsilon A; \quad R \tilde{R} = I \quad (C.1)$$

$\epsilon$  being an infinitesimal real number. Now

$$R \tilde{R} = (I + \epsilon A)(I + \epsilon \tilde{A}) = I + \epsilon(A + \tilde{A}) = I$$

$$\therefore \tilde{A} = -A \quad (C.2)$$

shows that matrix  $A$  must be antisymmetric so that one can write

$$\epsilon A_{ij} = \sum_{k=1}^3 \epsilon_{ijk} \epsilon_k \quad (C.3)$$

using the completely antisymmetric tensor  $\epsilon_{ijk}$  (see page 64).

Now, a function  $F(x_i) \equiv F(x_1, x_2, x_3)$  will transform to

$$F(x'_i) = F\left(\sum_j R_{ij} x_j\right)$$

$$F(x'_i) = F\left(x_i + \sum_{j,k} \epsilon_{ijk} \epsilon_k x_j\right)$$

$$= F(x_i) + \sum_{j,k} \epsilon_{ijk} \epsilon_k x_j \frac{\partial}{\partial x_i} F(x_i) + \dots \quad (C.4)$$

using the Taylor expansion in the last step. But the three cartesian angular momentum operators given explicitly in (39) can be written collectively as

$$L_k \equiv \frac{1}{i} \sum_{\ell f} \epsilon_{k\ell f} x_f \frac{\partial}{\partial x_\ell} \quad (C.5)$$

so that (4) now reads

$$F(x'_i) = F(x_i) - i \sum_{k=1}^3 \epsilon_k L_k F(x_i) \quad (C.6)$$

and an infinitesimal increment in  $F(x_i)$  due to an arbitrary rotation of the axes is

$$\delta F(x_i) \equiv F(x'_i) - F(x_i) = -i \sum_{k=1}^3 \epsilon_k L_k F(x_i). \quad (C.7)$$

Conclusion: The angular momentum component operators  $L_k$  ( $k=1,2,3$ ) which obey the cyclic commutation relations  $[L_1, L_2] = i L_3$  can be considered as the generators of an infinitesimal transformation of the three-dimensional rotation group  $R_3$ . The relations  $[L_1, L_2] = i L_3$  form a Lie algebra of the Lie group  $R_3$ .



# APPENDIX D.

## UNITARY GROUP GENERATORS.

In similar fashion, consider an infinitesimal unitary transformation acting in a t-dimensional space vector  $(x_1, x_2, \dots, x_t)$ . The vector-components will transform as

$$x'_l = \sum_j U_{lj} x_j \quad (l, j = 1, 2, \dots, t) \quad (D.1)$$

where by definition

$$\sum_j U_{lj} U_{kj}^* = \delta_{lk} \quad \text{or} \quad U U^\dagger = I \quad (D.2)$$

and

$$U = I + i\epsilon S \quad (D.3)$$

$\epsilon$  being a real, infinitesimal quantity and  $S^\dagger = S$  (hermitean) since

$$U U^\dagger = (I + i\epsilon S)(I - i\epsilon S^\dagger) \simeq I + i\epsilon(S - S^\dagger) = I. \quad (D.4)$$

An arbitrary function in this space  $F(x_i) \equiv F(x_1, x_2, \dots, x_t)$  will transform as

$$F(x'_l) = F\left(x_l + i\epsilon \sum_j S_{lj} x_j\right) \quad (D.5)$$

and using Taylor's theorem:

$$F(x'_l) = F(x_l) + i\epsilon \sum_{ij} \left( S_{ij} x_j \frac{\partial}{\partial x_i} F(x_l) \right) + \dots \quad (D.6)$$

To first order in  $\epsilon$ , the change in the function

$$\delta F(x_l) \equiv F(x'_l) - F(x_l) = i\epsilon \sum_{ij} \left( S_{ij} x_j \frac{\partial}{\partial x_i} \right) F(x_l) \quad (D.7)$$

is accomplished by  $t^2$  linearly independent operators

$$i \in \sum_{ij} S_{ij} x_j \frac{\partial}{\partial x_i} \quad (D.8)$$

where since  $S$  is an arbitrary hermitean matrix the  $t^2$  operators

$$x_j \frac{\partial}{\partial x_i} \equiv C_{ji} \quad (D.9)$$

could be taken as the generators for infinitesimal unitary transformations in the  $t$ -dimensional space. Their commutation relations, following from (D.9), are immediately seen to constitute the commutator algebra

$$[C_{ij}, C_{i'j'}] = C_{ij'} \delta_{i'i} - C_{i'j} \delta_{ij'} \quad (D.10)$$

identical in structure to (32) since here the metric  $g_{ij} = \delta_{ij}$  so that  $C_{ij} = C_i^j$ .

# APPENDIX E.

## ROTATIONAL GROUP GENERATORS.

In the same  $t$ -dimensional space spanned by the vector  $(x_1, x_2, \dots, x_t)$  consider the real orthogonal transformations

$$x'_i = \sum_j R_{ij} x_j \quad (i, j = 1, 2, \dots, t)$$

where by definition

$$\sum_j R_{ij} R_{ij} = \delta_{ii} \quad \text{or} \quad R \tilde{R} = I$$

with the added condition that  $\det |R_{ij}| = +1$  (rotations only, excluding reflections); moreover

$$R = I + \epsilon A$$

$\epsilon$  an infinitesimal real quantity and  $\tilde{A} = -A$  since

$$R \tilde{R} = (I + \epsilon A)(I + \epsilon \tilde{A}) = I + \epsilon(A + \tilde{A}) = I.$$

An arbitrary function in this space  $G(x) = G(x_1, x_2, \dots, x_t)$  will then transform like

$$G(x'_i) = G\left(x_i + \epsilon \sum_j A_{ij} x_j\right)$$

which on Taylor expansion becomes

$$G(x'_i) = G(x_i) + \epsilon \sum_{ij} A_{ij} x_j \frac{\partial}{\partial x_i} G(x_i) + \dots$$

$$G(x'_2) = G(x_2) + \epsilon \sum_{ij} A_{2j} C_{ji} G(x_2) + \dots$$

$C_{ji}$  being the operators (D.9) of Appendix D. Then

$$G(x'_2) = G(x_2) + 2\epsilon \sum_{i < j} A_{2j} \Lambda_{ji} G(x_2) + \dots$$

where we have called the operators

$$\Lambda_{ij} \equiv \frac{1}{2} (C_{ij} - C_{ji})$$

the generators of infinitesimal rotations in  $t$ -dimensions, of which there are  $\frac{1}{2} t(t-1)$  independent ones as the real antisymmetric matrix  $A$  has, for  $i < j$ ,  $\frac{1}{2} t(t-1)$  independent elements. In view of relations (10) of Appendix D one derives the relations

$$[\Lambda_{ij}, \Lambda_{i'j'}] = \frac{1}{2} (\Lambda_{ij'} \delta_{i'j} + \Lambda_{ji'} \delta_{ij} + \Lambda_{jj'} \delta_{ii'} + \Lambda_{i'i} \delta_{jj'})$$

which constitute the Lie algebra of the generators of the group  $R_t$ . The space metric here is again simply  $g_{ij} = \delta_{ij}$ .

# APPENDIX F.

## GROUP INVARIANCE OF AN OPERATOR.

Again, consider the  $t$ -dimensional linear vector space spanned by the vector  $\hat{x} \equiv (x_i) \equiv (x_1, x_2, \dots, x_t)$  which transforms unitarily as:

$$x_i = \sum_j U_{ji}^* x'_j. \quad (F.1)$$

An arbitrary function  $F(x_i) \equiv F(x_1, x_2, \dots, x_t)$  of the vector components will transform, let us say, as

$$\theta_U F(x_i) = F(x'_i), \quad \theta_U^\dagger \theta_U = I. \quad (F.2)$$

Consider an operator  $H$  for which  $F(x_i)$  is an eigenfunction, that is,

$$H F(x_i) = E F(x_i). \quad (F.3)$$

Then, providing that

$$[\theta_U, H] = 0 \quad \text{or} \quad \theta_U^\dagger H \theta_U = H \quad (F.4)$$

one has that the transformed function  $F(x'_i)$  is also an eigenfunction of  $H$ , since

$$\begin{aligned} H F(x_i) &= \theta_U^{-1} H \theta_U F(x_i) = E F(x_i) \\ \theta_U \theta_U^{-1} H \theta_U F(x_i) &= H \theta_U F(x_i) = E \theta_U F(x_i) \\ \therefore H F(x'_i) &= E F(x'_i), \end{aligned} \quad (F.5)$$

from which follows the Theorem: the group of  $t$ -dimensional unitary transformations  $U_t$  is the symmetry group of the operator  $H$  if  $H$  commutes with operation  $\mathcal{O}_U$ . (Common examples from both atomic and nuclear physics for  $\mathcal{O}_U$  are the permutation and rotation operations---in which cases the hamiltonian  $H$  is an invariant, respectively, of the permutation group  $\Pi_N$  and the rotation group  $R_3$ . Alternatively, these are the symmetry groups of  $H$ .)

Now, for a continuous group it is sufficient for it to be a symmetry group of  $H$  that  $\mathcal{O}_U$  is an infinitesimal transformation of that group namely, as in Appendix D,

$$\mathcal{O}_U \equiv (I + i\epsilon \sum_{ij} S_{ij} \mathcal{G}_{ji}) \quad (F.6)$$

recalling that  $S_{ij}$  is an arbitrary hermitean matrix and  $\mathcal{G}_{ji}$  the  $t^2$  generators of infinitesimal transformations of the group  $U_t$ .

Hence if (4) is fulfilled it follows that

$$[H, \mathcal{G}_{ji}] = 0.$$

Conclusion: A necessary and sufficient condition for the invariance of an operator  $H$  under a group  $U_t$  is that  $H$  commute with the group generators.

Consider two different, complete, linearly independent sets of functions of  $\lambda_i$  ( $i=1,2,\dots,t$ ),

$$\psi_m^{(r)} \equiv \{\psi_1^{(r)}, \psi_2^{(r)}, \dots, \psi_{n_r}^{(r)}\} \quad (F.8)$$

$$\phi_m^{(s)} \equiv \{\phi_1^{(s)}, \phi_2^{(s)}, \dots, \phi_{n_s}^{(s)}\} \quad (F.9)$$

which are basis for two distinct irreducible representations labelled by  $j$  and  $j'$ , with rows by  $m$  and  $m'$ , respectively, of a group  $U_t$ . Since  $\mathcal{O}_U$  is unitary, the scalar product

$$(\psi_m^{(j)}, \phi_{m'}^{(j')}) = (\mathcal{O}_U \psi_m^{(j)}, \mathcal{O}_U \phi_{m'}^{(j')}). \quad (F.10)$$

But, by definition of an irreducible basis,<sup>74)</sup>

$$\mathcal{O}_U \psi_m^{(j)} = \sum_k \psi_k^{(j)} D_{km}^{(j)}(u) \quad (F.11)$$

$$\mathcal{O}_U \phi_{m'}^{(j')} = \sum_{k'} \phi_{k'}^{(j')} D_{k'm'}^{(j')}(u) \quad (F.12)$$

the  $D_{km}^{(j)}(u)$  being the irreducible representation  $j$  with rows and columns  $k, m$  and  $u$  refers to the elements of the group  $U_t$ .

Likewise for  $D_{k'm'}^{(j')}(u)$ . These obey the well-known orthonormality relation<sup>74)</sup>

$$\sum_u D_{km}^{(j)*}(u) D_{k'm'}^{(j')}(u) = \frac{h}{n_j} \delta_{jj'} \delta_{kk'} \delta_{mm'} \quad (F.13)$$

$h$  being the number of elements in the group (which is irrelevant for our present purposes) and from (8)  $n_j$  is the dimensionality of  $D_{km}^{(j)}(u)$ . Thus (F.10) becomes

$$(\psi_m^{(j)}, \phi_{m'}^{(j')}) = \sum_{kk'} D_{km}^{(j)*}(u) D_{k'm'}^{(j')}(u) (\psi_k^{(j)}, \phi_{k'}^{(j')}) \quad (F.14)$$

which upon summation over all the elements,  $h$  appearing as a factor on both sides, one obtains

$$(\psi_m^{(j)}, \phi_{m'}^{(j')}) = \frac{1}{n_j} \delta_{jj'} \delta_{mm'} \sum_k (\psi_k^{(j)}, \phi_k^{(j')}). \quad (F.15)$$

Remembering that, by stipulation, the operator  $H$  commutes with  $\mathcal{O}_U$  we have

$$H \mathcal{O}_v \psi_m^{(j)} = \mathcal{O}_v (H \psi_m^{(j)}) = \sum_k (H \psi_k^{(j)}) \mathcal{D}_{km}^{(j)}(u) \quad (F.16)$$

wherefrom clearly  $H \psi_m^{(j)}$  is an irreducible basis if  $\psi_m^{(j)}$  is such a one, and thus the result (15) can be applied to give

$$(\phi_{m'}^{(j)}, H \psi_m^{(j)}) = \frac{1}{n_j} \delta_{jj'} \delta_{mm'} \sum_k (\phi_m^{(j)}, H \psi_m^{(j)}) \quad (F.17)$$

which is one of the most useful results of the theory of group representations, it states:

Conclusion: An operator  $H$  having  $U_t$  as a symmetry group will have non-zero matrix elements only between states of a given irreducible representation  $j$  of  $U_t$  and furthermore these elements are independent of the row  $m$  of that representation.



APPENDIX G.

THE COEFFICIENTS  $\langle n'_1 n'_2 n'_0 | C_q^{q'} | n_1 n_2 n_0 \rangle$ .

The coefficients appearing in the  $U_3$  group generator expansion ( 133 ) are easily found. The annihilation boson operator  $a^{q'}$  becomes a differential operator ( 124 ). Then, in view of ( 119 )

$$C_q^{q'} |n_1 n_2 n_0\rangle = a_q^+ \frac{\partial}{\partial a_q^+} \frac{(a_1^+)^{n_1} (a_2^+)^{n_2} (a_0^+)^{n_0}}{\sqrt{n_1! n_2! n_0!}} |\bar{0}\rangle$$

$|\bar{0}\rangle$  referring to the ground state in our tri-dimensional harmonic oscillator. Differentiating and multiplying by  $a_q^+$  one gets

$$C_q^{q'} |n_1 n_2 n_0\rangle = n_{q'} \frac{(a_1^+)^{n_1 - \delta_{q'1} + \delta_{q1}} (a_2^+)^{n_2 - \delta_{q'2} + \delta_{q2}} (a_0^+)^{n_0 - \delta_{q'0} + \delta_{q0}}}{\sqrt{n_1! n_2! n_0!}} |\bar{0}\rangle$$

Thus

$$\begin{aligned} \langle n'_1 n'_2 n'_0 | C_q^{q'} | n_1 n_2 n_0 \rangle &= \\ &= \langle n_1 - \delta_{q'1} + \delta_{q1}, n_2 - \delta_{q'2} + \delta_{q2}, n_0 - \delta_{q'0} + \delta_{q0} | C_q^{q'} | n_1 n_2 n_0 \rangle \\ &= n_{q'} \sqrt{\frac{(n_1 - \delta_{q'1} + \delta_{q1})! (n_2 - \delta_{q'2} + \delta_{q2})! (n_0 - \delta_{q'0} + \delta_{q0})!}{n_1! n_2! n_0!}} \\ &= \sqrt{n_{q'} (n_{q'} - \delta_{q'q} + 1)} . \end{aligned}$$

# APPENDIX H.

## MATRIX ELEMENTS OF $U_3$ GENERATORS:

Calculation of the matrix elements of the generators of a unitary group  $U_r$  in a base characterized by the canonical chain  $U_r \supset U_{r-1} \supset \dots \supset U_2 \supset U_1$ , has been discussed by Gelfand & Zetlin,<sup>42)</sup> Biedenharn<sup>75)</sup> and Moshinsky.<sup>45)</sup> We here simply list the results for the 9  $U_3$  generators  $C_q^{(q)}$  in the chain  $U_3 \supset U_2 \supset U_1$ , in a manner useful for calculations ( cf. Section on Spin Orbit Force for notation). The labels  $(k_1, k_2)$  in the bras and kets, being redundant, are suppressed.

$$\begin{aligned} \langle q_1, q_2, M | C_1^1 | q_1, q_2, M \rangle &= w_1 = \frac{1}{2}(M + q_1 + q_2) \\ \langle q_1, q_2, M | C_2^2 | q_1, q_2, M \rangle &= w_2 = \frac{1}{2}(q_1 + q_2 - M) \\ \langle q_1, q_2, M | C_3^3 | q_1, q_2, M \rangle &= w_3 = k_1 + k_2 - q_1 - q_2 \end{aligned} \quad [q, q' = 1, \bar{1}, 0 \rightarrow 1, 2, 3]$$

$$\langle q_1', q_2', M' | C_1^2 | q_1, q_2, M \rangle = \frac{1}{2} \sqrt{(q_1 - q_2 + M + 2)(q_1 - q_2 - M)} \delta_{q_1', q_1} \delta_{q_2', q_2} \delta_{M', M+2}$$

$$\langle q_1', q_2', M' | C_2^1 | q_1, q_2, M \rangle = \frac{1}{2} \sqrt{(q_1 - q_2 + M)(q_1 - q_2 - M + 2)} \delta_{q_1', q_1} \delta_{q_2', q_2} \delta_{M', M-2}$$

$$\langle q_1' q_2' M' | C_1^3 | q_1 q_2 M \rangle =$$

$$\sqrt{\frac{(k_1 - q_1)(q_1 - k_2 + 1)(q_1 + 2)(q_1 - q_2 + M + 2)}{2(q_1 - q_2 + 1)(q_1 - q_2 + 2)}} \delta_{q_1' q_1 + 1} \delta_{q_2' q_2} \delta_{M' M + 1}$$

$$- \sqrt{\frac{(q_2 + 1)(k_2 - q_2)(k_1 - q_2 + 1)(q_1 - q_2 - M)}{2(q_1 - q_2)(q_1 - q_2 + 1)}} \delta_{q_1' q_1} \delta_{q_2' q_2 + 1} \delta_{M' M + 1}$$

$$\langle q_1' q_2' M' | C_3^1 | q_1 q_2 M \rangle =$$

$$\sqrt{\frac{(k_1 - q_1 + 1)(q_1 - k_2)(q_1 + 1)(q_1 - q_2 + M)}{2(q_1 - q_2)(q_1 - q_2 + 1)}} \delta_{q_1' q_1 - 1} \delta_{q_2' q_2} \delta_{M' M - 1}$$

$$- \sqrt{\frac{(q_2)(k_2 - q_2 + 1)(k_1 - q_2 + 2)(q_1 - q_2 - M + 2)}{2(q_1 - q_2 + 1)(q_1 - q_2 + 2)}} \delta_{q_1' q_1} \delta_{q_2' q_2 - 1} \delta_{M' M - 1}$$

$$\langle q_1' q_2' M' | C_2^3 | q_1 q_2 M \rangle =$$

$$\sqrt{\frac{(k_1 - q_1)(q_1 - k_2 + 1)(q_1 + 2)(q_1 - q_2 - M + 2)}{2(q_1 - q_2 + 2)(q_1 - q_2 + 1)}} \delta_{q_1' q_1 + 1} \delta_{q_2' q_2} \delta_{M' M - 1}$$

$$+ \sqrt{\frac{(q_2 + 1)(k_2 - q_2)(k_1 - q_2 + 1)(q_1 - q_2 + M)}{2(q_1 - q_2 + 1)(q_1 - q_2)}} \delta_{q_1' q_1} \delta_{q_2' q_2 + 1} \delta_{M' M - 1}$$

$$\langle q_1' q_2' M' | C_3^2 | q_1 q_2 M \rangle =$$

$$\sqrt{\frac{(k_1 - q_1 + 1)(q_1 - k_2)(q_1 + 1)(q_1 - q_2 - M)}{2(q_1 - q_2 + 1)(q_1 - q_2)}} \delta_{q_1' q_1 - 1} \delta_{q_2' q_2} \delta_{M' M + 1}$$

$$+ \sqrt{\frac{q_2(k_2 - q_2 + 1)(k_1 - q_2 + 2)(q_1 - q_2 + M + 2)}{2(q_1 - q_2 + 2)(q_1 - q_2 + 1)}} \delta_{q_1' q_1} \delta_{q_2' q_2 - 1} \delta_{M' M + 1}$$

# APPENDIX I.

## TRANSFORMATION BETWEEN $U_3 \supset U_2 \supset U_1$ AND $U_3 \supset R_3 \supset R_2$ BASES.

Both the basis set  $|(k, k_2) q, q_2 M_L\rangle$  transforming irreducibly under  $U_3 \supset U_2 \supset U_1$  and the set  $|(k, k_2) \omega L M_L\rangle$  irreducible under  $U_3 \supset R_3 \supset R_2$  constitute complete sets of normalizable wave functions in second-quantization formalism. Thus one may expand

$$|(k, k_2) \omega L M_L\rangle = \sum_{q, q_2} \langle q, q_2 (k, k_2 M_L) \omega L \rangle |(k, k_2) q, q_2 M_L\rangle \quad (I.1)$$

where  $\langle q, q_2 (k, k_2 M_L) \omega L \rangle$  denote (orthogonal) transformation coefficients.<sup>66)</sup> Calculation of matrix elements in the canonical chain  $U_3 \supset U_2 \supset U_1$  is simpler than directly in  $U_3 \supset R_3 \supset R_2$  as construction of the base in the former scheme is more direct, as was seen in Chapter VI. However, the latter chain is the physically significant one as it provides the quantum number L. Coefficients (I.1) are thus needed.

We know that

$$L^2 |(k, k_2) \omega L M_L\rangle = L(L+1) |(k, k_2) \omega L M_L\rangle \quad (I.2)$$

but, in general,

$$L^2 |(k, k_2) q, q_2 M_L\rangle = \sum_{q', q'_2} \langle (k, k_2) q', q'_2 M_L | L^2 |(k, k_2) q, q_2 M_L \rangle |(k, k_2) q', q'_2 M_L \rangle \quad (I.3)$$

Equations (1), (2) and (3) combined give us

$$\sum_{\substack{q_1, q_2 \\ q'_1, q'_2}} \langle q_1, q_2(k, k_2, M_L) \omega_L \rangle \langle (k, k_2) q'_1 q'_2 M_L | \mathcal{L}^2 | (k, k_2) q_1 q_2 M_L \rangle | (k, k_2) q'_1 q'_2 M_L \rangle \\ = L(L+1) \sum_{q_1, q_2} \langle q_1, q_2(k, k_2, M_L) \omega_L \rangle | (k, k_2) q_1 q_2 M_L \rangle. \quad (\text{I.4})$$

Multiplying both sides by  $\langle (k, k_2) q''_1 q''_2 M_L |$  and summing over  $q''_1 q''_2$ , supposing set  $| (k, k_2) q''_1 q''_2 M_L \rangle$  to be normalized, one gets

$$\sum_{q_1, q_2} \langle q_1, q_2(k, k_2, M_L) \omega_L \rangle \langle (k, k_2) q''_1 q''_2 M_L | \mathcal{L}^2 | (k, k_2) q_1 q_2 M_L \rangle \\ = L(L+1) \langle q''_1 q''_2(k, k_2, M_L) \omega_L \rangle$$

and multiplying by the (transposed) coefficients  $\langle \omega' L' (k, k_2, M_L) q''_1 q''_2 \rangle$  and summing again over  $q''_1 q''_2$  we obtain the result:

$$\sum_{\substack{q_1, q_2 \\ q''_1, q''_2}} \langle q_1, q_2(k, k_2, M_L) \omega_L \rangle \langle (k, k_2) q''_1 q''_2 M_L | \mathcal{L}^2 | (k, k_2) q_1 q_2 M_L \rangle \langle \omega' L' (k, k_2, M_L) q''_1 q''_2 \rangle \\ = L(L+1) \sum_{q''_1, q''_2} \langle q''_1 q''_2(k, k_2, M_L) \omega_L \rangle \langle \omega' L' (k, k_2, M_L) q''_1 q''_2 \rangle \\ = L(L+1) \delta_{\omega \omega'} \delta_{L L'} \quad (\text{I.5})$$

$$\|S\|_X \|\mathcal{L}^2\|_X \|\tilde{S}\| = L(L+1) I$$

meaning that the desired coefficients  $\langle q_1, q_2 (k, k_2, M_L) \omega_L \rangle$  are nothing more than the eigenvector components of the matrix representation of operator  $\mathcal{L}^2$  in the base irreducible under  $U_3 \supset U_2 \supset U_1$ .

To obtain them, we can construct the matrix  $\| \langle (k, k_2) q_1' q_2' M_L | \mathcal{L}^2 | (k, k_2) q_1 q_2 M_L \rangle \|$ , diagonalize it by computer and obtain at the same time the associated eigenvectors. The construction is simplified since  $\mathcal{L}^2$  by (141) is expressible bilinearly in terms of  $U_3$  group generators  $C_q^{q'}$  whose matrix elements in the scheme  $U_3 \supset U_2 \supset U_1$  are known (Appendix H). We know from (171) that

$$\begin{aligned} \mathcal{L}^2 &= \sum_q (-)^q \mathcal{L}_q \mathcal{L}_{-q} \quad (q = 1, \bar{1}, 0) \\ &= -2 \mathcal{L}_{\bar{1}} \mathcal{L}_1 + \mathcal{L}_0 (\mathcal{L}_0 + 1) \end{aligned}$$

if use is made of  $R_3$  commutation relations (142b). Suppressing labels  $(k, k_2)$  we have

$$\begin{aligned} \langle q_1' q_2' M_L | \mathcal{L}^2 | q_1 q_2 M_L \rangle &= \\ -2 \langle q_1' q_2' M_L | \mathcal{L}_{\bar{1}} \mathcal{L}_1 | q_1 q_2 M_L \rangle &+ M_L (M_L + 1) \delta_{q_1' q_1} \delta_{q_2' q_2} \end{aligned}$$

since  $\mathcal{L}_0$  is diagonal in  $U_3 \supset U_2 \supset U_1$  by (142a), (236) and (237). Now, since  $\mathcal{L}_{\bar{1}}^\dagger = -\mathcal{L}_1$

$$\begin{aligned} \langle q_1' q_2' M_L | \mathcal{L}^2 | q_1 q_2 M_L \rangle &= \\ = 2 \sum_{q_1'' q_2''} \langle q_1'' q_2'' M_L | \mathcal{L}_1 | q_1' q_2' M_L \rangle^* \langle q_1'' q_2'' M_L | \mathcal{L}_{\bar{1}} | q_1 q_2 M_L \rangle &+ M_L (M_L + 1) \delta_{q_1' q_1} \delta_{q_2' q_2} \end{aligned}$$

$$\begin{aligned} \langle q'_1 q'_2 M_L | \mathcal{L}^2 | q_1 q_2 M_L \rangle = \\ 2 \sum_{q''_1 q''_2} \langle q''_1 q''_2 M_L | C_1^3 + C_3^2 | q'_1 q'_2 M_L \rangle \langle q''_1 q''_2 M_L | C_1^3 + C_3^2 | q_1 q_2 M_L \rangle \\ + M_L(M_L+1) \delta_{q'_1 q_1} \delta_{q'_2 q_2} \end{aligned} \quad (I.6)$$

since  $\langle | \mathcal{L}_1 | \rangle$  are real and from (142a),  $\mathcal{L}_1 = -(C_1^3 + C_3^2)$ . To evaluate (6) in general form one needs essentially the algebraic expressions of matrix elements

$$\langle q'_1 q'_2 M_L | C_1^3 | q_1 q_2 M_L \rangle \quad \text{and} \quad \langle q'_1 q'_2 M_L | C_3^2 | q_1 q_2 M_L \rangle$$

which are included in the list of Appendix H. Carrying out the calculation one obtains the formula in the following page. Only one of the seven terms with double deltas will contribute to a given element of the matrix so that it is relatively simple to use.

The matrix  $\| \langle (k, k_2) q'_1 q'_2 M_L | \mathcal{L}^2 | (k, k_2) q_1 q_2 M_L \rangle \|$  is formed in blocks or submatrices labeled by  $M_L$  and with rows and columns given by  $q_1 q_2$  where  $M_L$  refers to the positive projections of all the  $L$  values contained in the given  $(k, k_2)$  representation of interest (See Table V.4.A). For  $(k, k_2) = (71)$ , for instance,  $L = 1, 2, 3, 4, 5, 6, 7$  so that

$$M_L = 0^7, 1^7, 2^6, 3^5, 4^4, 5^3, 6^2, 7.$$

Our formula is:

$$\langle (k_1 k_2) q_1 q_2 M_L | \mathcal{L}^2 | (k_1 k_2) q'_1 q'_2 M_L \rangle =$$

$$\begin{aligned} & \delta_{q'_1 q_1} \delta_{q'_2 q_2} \left\{ M_L(M_L+1) + \frac{(q_1 - q_2 - M_L)}{(q_1 - q_2)(q_1 - q_2 + 1)} \left[ (k_1 - q_2)(k_1 - q_2 + 1)(q_2 + 1) + (k_1 - q_1 + 1)(q_1 - k_2)(q_1 + 1) \right] \right. \\ & \quad \left. + \frac{(q_1 - q_2 + M_L + 2)}{(q_1 - q_2 + 2)(q_1 - q_2 + 1)} \left[ (k_1 - q_1)(q_1 - k_2 + 1)(q_1 + 2) + (k_2 - q_2 + 1)(q_1 + 2) + (k_1 - q_2 + 2)q_2 \right] \right\} \\ & + \delta_{q'_1 q_1 + 1} \delta_{q'_2 q_2 + 1} \left[ \frac{\sqrt{(k_1 - q_1)(q_1 - k_2 + 1)(q_1 + 2)(k_2 - q_2)(k_1 - q_2 + 1)(q_2 + 1)}}{(q_1 - q_2 + 1)} \right. \\ & \quad \left. - \frac{\left[ \frac{(q_1 - q_2 + M_L + 2)}{(q_1 - q_2 + 2)} - \frac{(q_1 - q_2 - M_L)}{(q_1 - q_2)} \right]}{\left[ \frac{(q_1 - q_2 + M_L + 2)}{(q_1 - q_2 + 2)} - \frac{(q_1 - q_2 - M_L)}{(q_1 - q_2)} \right]} \right] \\ & + \delta_{q'_1 q_1 - 1} \delta_{q'_2 q_2 - 1} \left[ \frac{\sqrt{(k_2 - q_2 + 1)(k_1 - q_2 + 2)q_2(k_1 - q_1 + 1)(q_1 - k_2)(q_1 + 1)}}{(q_1 - q_2 + 1)} \right. \\ & \quad \left. - \frac{\left[ \frac{(q_1 - q_2 + M_L + 2)}{(q_1 - q_2 + 2)} - \frac{(q_1 - q_2 - M_L)}{(q_1 - q_2)} \right]}{\left[ \frac{(q_1 - q_2 + M_L + 2)}{(q_1 - q_2 + 2)} - \frac{(q_1 - q_2 - M_L)}{(q_1 - q_2)} \right]} \right] \\ & + \delta_{q'_1 q_1 + 2} \delta_{q'_2 q_2} \sqrt{\frac{(k_1 - q_1)(q_1 - k_2 + 1)(q_1 + 2)(q_1 - q_2 + M_L + 2)(k_1 - q_1 - 1)(q_1 - k_2 + 2)(q_1 + 3)(q_1 - q_2 - M_L + 2)}{(q_1 - q_2 + 1)(q_1 - q_2 + 2)^2 (q_1 - q_2 + 3)}} \\ & + \delta_{q'_1 q_1 - 2} \delta_{q'_2 q_2} \sqrt{\frac{(k_1 - q_1 + 1)(q_1 - k_2)(q_1 + 1)(q_1 - q_2 - M_L)(k_1 - q_1 + 2)(q_1 - k_2 - 1)q_1(q_1 - q_2 + M_L)}{(q_1 - q_2 + 1)(q_1 - q_2)^2 (q_1 - q_2 - 1)}} \\ & - \delta_{q'_1 q_1} \delta_{q'_2 q_2 + 2} \sqrt{\frac{(k_2 - q_2)(k_1 - q_2 + 1)(q_2 + 1)(q_1 - q_2 - M_L)(k_2 - q_2 - 1)(k_1 - q_2)(q_2 + 2)(q_1 - q_2 + M_L)}{(q_1 - q_2 + 1)(q_1 - q_2)^2 (q_1 - q_2 - 1)}} \\ & - \delta_{q'_1 q_1} \delta_{q'_2 q_2 - 2} \sqrt{\frac{(k_2 - q_2 + 1)(k_1 - q_2 + 2)q_2(q_1 - q_2 + M_L + 2)(k_2 - q_2 + 2)(k_1 - q_2 + 3)(q_2 - 1)(q_1 - q_2 - M_L + 2)}{(q_1 - q_2 + 1)(q_1 - q_2 + 2)^2 (q_1 - q_2 + 3)}} \end{aligned}$$

(I.7)



where it is to be noted the the sum of block dimensions equals the dimensionality of  $P_{q_1 q_2 M_L}^{(71)}$  for  $M_L \geq 0$  (page 176), i.e., 35.

Our calculations will however only require one submatrix (see chapter VIII), that for  $M=1$ , e.g., which is 7 X 7 and is associated with all seven values  $L = 1, 2, \dots, 7$ . Using formula on page 229 one gets

$$\| \langle (71) q_1 q_2 M_L=1 | \mathcal{L}^2 | (71) q'_1 q'_2 M_L=1 \rangle \| =$$

$q'_1 q'_2 \backslash q_1 q_2$	61	41	21	70	50	30	10
61	22	$12\sqrt{\frac{7}{5}}$	0	0	$\frac{8}{\sqrt{35}}$	0	0
41	$12\sqrt{\frac{7}{5}}$	36	$20\sqrt{\frac{2}{3}}$	0	0	$8\sqrt{\frac{2}{15}}$	0
21	0	$20\sqrt{\frac{2}{3}}$	34	0	0	0	8
70	0	0	0	8	$4\sqrt{\frac{30}{7}}$	0	0
50	$\frac{8}{\sqrt{35}}$	0	0	$4\sqrt{\frac{30}{7}}$	26	$12\sqrt{\frac{6}{5}}$	0
30	0	$8\sqrt{\frac{2}{15}}$	0	0	$12\sqrt{\frac{6}{5}}$	28	$4\sqrt{5}$
10	0	0	8	0	0	$4\sqrt{5}$	14

which must fulfill the requirement

$$\text{Tr} \| \langle (k, k_2) q_1 q_2 M_L | \mathcal{L}^2 | (k, k_2) q_1' q_2' M_L \rangle \| = \sum_L L(L+1) \quad (\text{I.8})$$

the sum on the right extending over all L-values contained in  $(k, k_2)$ . The normalized eigenvector components obtained by computer diagonalization of above matrix are:

$$\| \langle q_1 q_2 (71 M_L=1) \omega L \rangle \| = \quad (\text{I.9})$$

$\begin{smallmatrix} L \\ q_1 q_2 \end{smallmatrix}$	1	2	3	4	5	6	7
61	0.180702	-0.269374	0.645553	-0.163451	-0.596548	-0.064210	0.302220
41	-0.213809	0.318728	-0.462317	-0.032232	-0.321474	-0.151947	0.715184
21	0.218218	-0.325300	0.164122	0.263181	0.627675	-0.124064	0.583945
70	0.590169	0.659829	0.166450	0.400371	-0.057870	0.157280	0.015423
50	-0.427618	-0.159364	0.080403	0.580193	-0.153748	0.645777	0.089398
30	0.390360	-0.145479	-0.256892	-0.500216	0.051037	0.693542	0.163218
10	-0.436436	0.487950	0.492366	-0.394771	0.342368	0.186097	0.145986

The label  $\omega$  is ignored as no multiple L-values appear under  $(k, k_2) = (71)$  (See Table V.11).

APPENDIX J.

QUADRUPOLE MOMENT AND TRANSITION RATE OPERATORS IN SECOND-  
QUANTIZATION FORMULISM.

The mass quadrupole moment of a given N-particle state is defined<sup>76)</sup> as the expectation value of the operator

$$Q_c \equiv \sum_i^N (3z_i^2 - r_i^2) \quad (J.1)$$

which is of the single-body type (23a). Being spin-isospin independent, (J.1) cast into second-quantization language by (35) becomes

$$Q_c = \sum_{\mu\mu'} \langle \mu | 3z^2 - r^2 | \mu' \rangle b_{\mu}^{\mu'} \quad (J.2)$$

where  $\mu \equiv (n, n_1, n_0)$  denote quantum numbers of a single particle in a harmonic oscillator common potential and  $\{b_{\mu}^{\mu'}\}$  the set of  $\mathcal{U}_r$  generators discussed before. Restriction of states to a single harmonic oscillator mayor shell means  $\nu = n_1 + n_1' + n_0 = n_1' + n_1' + n_0$  and moreover that

$$\begin{aligned} \langle \mu | 3z^2 - r^2 | \mu' \rangle &= \frac{1}{2} \langle \mu | 3z^2 - r^2 + 3p_z^2 - p^2 | \mu' \rangle \\ &= \langle \mu | 3C_0^z - \sum_q C_q^z - \frac{3}{2} | \mu' \rangle \end{aligned}$$

because of definitions (113) and (117). From (125) we know that  $C_q^z | \mu \rangle \equiv C_q^z | n, n_1, n_0 \rangle = n_q | n, n_1, n_0 \rangle$  for  $q = 1, \bar{1}, 0$  so that the desired matrix elements

$$\langle \mu | 3z^2 - r^2 | \mu' \rangle = [2n_0 - (n_1 + n_2) - \frac{3}{2}] \delta_{\mu\mu'}$$

$$\delta_{\mu\mu'} \equiv \delta_{m_1 n_1} \delta_{m_2 n_2} \delta_{m_0 n_0}.$$

The mass quadrupole operator  $Q_0$  in terms of  $U_{\nu=6}$  generators is thus very simply

$$Q_0 = -\frac{7}{2} (b_1' + b_2'^2 + b_4') - \frac{1}{2} (b_3^3 + b_5^5) + \frac{5}{2} b_6^6 \quad (J.3)$$

whose effect on states of the type  $\Delta_\mu' \Delta_{\mu_1 \mu_2 \mu_3}^{123}$  is clear from (304).

The transition rate for quadrupole de-excitations will involve<sup>77)</sup> the operator  $Q_2 = \sum_i x_{+i}^2 \equiv \sum_i (x + iy)_i^2$  which here becomes

$$Q_2 = \sum_{\mu\mu'} \langle \mu | x_{+}^2 | \mu' \rangle b_{\mu}^{\mu'} \quad (J.4)$$

Again, restriction to a single oscillator shell  $\nu$  implies

$$\begin{aligned} \langle \mu | x_{+}^2 | \mu' \rangle &= \frac{1}{2} \langle \mu | x_{+}^2 + p_{+}^2 | \mu' \rangle \\ &= \langle \mu | C_1' + \frac{3}{2} | \mu' \rangle \\ &= (n_1 + \frac{3}{2}) \delta_{\mu\mu'} \end{aligned}$$

and therefore our operator (J.4) is simply

$$Q_2 = \frac{7}{2} b_1' + \frac{5}{2} (b_2'^2 + b_3^3), \quad (J.5)$$

of the same type as  $Q_0$  of (J.3).

The low-lying states, e.g., of  $F^{20}$  are designated by  $| [31](71) J M_J = J, T=1 \rangle$  which will be linear combinations of terms  $\Delta_\mu' \Delta_{\mu_1 \mu_2 \mu_3}^{123}$  of  $U_6 \supset U_3$  symmetry  $[f](k_1 k_2) = [31](71)$ .

Enhancement of calculated quadrupole moments and transition rates due to collective core effects can be estimated by the addition to each nucleon of a certain fraction of the proton charge  $e$  ( $\sim 0.5e$  for  $O^{17}$ ).<sup>7, 22)</sup>

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## KEY WORDS

## LINK A

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2s-1d nuclei

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